

# An elliptic equation with spike solutions concentrating at local minima of the Laplacian of the potential \*

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## Abstract

We consider the equation  $-\epsilon^2 \Delta u + V(z)u = f(u)$  which arises in the study of nonlinear Schrödinger equations. We seek solutions that are positive on  $\mathbb{R}^N$  and that vanish at infinity. Under the assumption that  $f$  satisfies super-linear and sub-critical growth conditions, we show that for small  $\epsilon$  there exist solutions that concentrate near local minima of  $V$ . The local minima may occur in unbounded components, as long as the Laplacian of  $V$  achieves a strict local minimum along such a component. Our proofs employ variational mountain-pass and concentration compactness arguments. A penalization technique developed by Felmer and del Pino is used to handle the lack of compactness and the absence of the Palais-Smale condition in the variational framework.

## 1 Introduction

This paper concerns the equation

$$-\epsilon^2 \Delta u + V(z)u = f(u) \tag{1.1}$$

on  $\mathbb{R}^N$  with  $N \geq 1$ , where  $f(u)$  is a “superlinear” type function such as  $f(u) = u^p$ ,  $p > 1$ . Such an equation arises when searching for standing wave solutions of the nonlinear Schrödinger equation (see [3]). For small positive  $\epsilon$ , we seek “ground states,” that is, positive solutions  $u$  with  $u(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ . Floer and Weinstein ([6]) examined the case  $N = 1$ ,  $f(u) = u^3$  and found that for small  $\epsilon$ , a ground state  $u_\epsilon$  exists which concentrates near a non-degenerate critical point of  $V$ . Similar results for  $N > 1$  were obtained by Oh in [10]-[12]. In [3], del Pino and Felmer found that if  $V$  has a strict local minimum, then for small  $\epsilon$ , (1.1) has a ground state concentrating near that minimum. A strict local minimum occurs when there exists a bounded, open set  $\Lambda \subset \mathbb{R}^N$

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with  $\inf_{\Lambda} V < \inf_{\partial\Lambda} V$ . They extended their results in [4] to the more general case where  $V$  has a “topologically stable” critical point, that is, a critical point obtained via a topological linking argument (see [4] for a precise formulation). Such a critical point persists under small perturbations of  $V$ . Examples are a strict local extremum and a saddle point. This very strong result is notable because the critical points of  $V$  in question need not be non-degenerate or even isolated. Similar results have been obtained by Li [8], and earlier work of Rabinowitz [13] is also interesting. The recent results of [1] and [9] also permit  $V$  to have degenerate critical points.

A common feature of all the papers above is that  $V$  must have a non-degenerate, or at least topologically stable, set of critical points. Therefore it is natural to try to remove this requirement. That we must assume some conditions on  $V$  is shown by Wang’s counterexample [15] - if  $V$  is nondecreasing and nonconstant in one variable (e.g.  $V(x_1, x_2, x_3) = 2 + \tan^{-1}(x_1)$ ), then no ground states exist. In [14] the author showed that ground states to (1.1) exist under the assumption that  $V$  is almost periodic, together with another mild assumption. Those assumptions did not guarantee that  $V$  had a topologically stable critical point.

Aside from periodicity or recurrence properties of  $V$ , another approach is to impose conditions on the derivatives of  $V$ . That is the approach taken here. We will assume that  $V$  has a (perhaps unbounded) component of local minima, along which  $\Delta V$  achieves a strict local minimum. More specifically, assume  $f$  satisfies the following:

$$(F1) \quad f \in C^1(\mathbb{R}^+, \mathbb{R})$$

$$(F2) \quad f'(0) = 0 = f(0).$$

$$(F3) \quad \lim_{q \rightarrow \infty} f(q)/q^s = 0 \text{ for some } s > 1, \text{ with } s < (N + 2)/(N - 2) \text{ if } N \geq 3.$$

$$(F4) \quad \text{For some } \theta > 2, 0 < \theta F(q) \leq f(q)q \text{ for all } q > 0, \text{ where } F(\xi) \equiv \int_0^\xi f(t) dt.$$

$$(F5) \quad \text{The function } q \mapsto f(q)/q \text{ is increasing on } (0, \infty).$$

Assumptions (F1)-(F5) are the same as in [3] and are satisfied by  $f(q) = q^s$ , for example, if  $1 < s < (N + 2)/(N - 2)$ . Assume that  $V$  satisfies the following:

$$(V1) \quad V \in C^2(\mathbb{R}^n, \mathbb{R})$$

$$(V2) \quad D^\alpha V \text{ is bounded and Lipschitz continuous for } |\alpha| = 2.$$

$$(V3) \quad 0 < V_- \equiv \inf_{\mathbb{R}^N} V < \sup_{\mathbb{R}^N} V \equiv V^+ < \infty$$

$$(V4) \quad \text{There exists a bounded, nonempty open set } \Lambda \subset \mathbb{R}^N \text{ and a point } z_0 \in \Lambda \text{ with } V(z_0) = \inf_{\Lambda} V \equiv V_0, \text{ and}$$

$$\Delta_0 \equiv \inf\{\Delta V(z) \mid z \in \Lambda, V(z) = V_0\} < \inf\{\Delta V(z) \mid z \in \partial\Lambda, V(z) = V_0\}$$

**Note:** A special case of (V4) occurs when  $\Lambda$  is bounded and  $V(z_0) < \inf_{\partial\Lambda} V$ ; this case is treated, under weaker hypotheses, in [3]. A specific example of (V4) is if  $N = 2$  and  $V$  satisfies (V1)-(V4), with  $V(z_1, z_2) = 1 + (z_1^2 - z_2)^2$  for  $z_1^2 + z_2^2 \leq 1$ . Then  $\Delta V(z_1, z_1^2) = 8z_1^2 + 2$  for  $z_1^2 + z_2^2 \leq 1$ , so we may take  $\Lambda = B_1(0, 0) \subset \mathbb{R}^2$  and  $z_0 = (0, 0)$ . Then  $V$  has a component of local minima that includes the parabolic arc  $\{z_2 = z_1^2\} \cap B_1(0, 0)$ , along which  $\Delta V$  has a minimum of 2 at  $(0, 0)$ , with  $\Delta V > 2$  at the two endpoints of the arc.

We prove the following:

**Theorem 1.1** *Let  $V$  and  $f$  satisfy (V1)-(V4) and (F1)-(F5). Then there exists  $\epsilon_0 > 0$  such that if  $\epsilon \leq \epsilon_0$ , then (1.0) has a positive solution  $u_\epsilon$  with  $u_\epsilon(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ .  $u_\epsilon$  has exactly one local maximum (hence, global maximum) point  $z_\epsilon \in \Lambda$ , where  $\Lambda$  is as in (V4). There exist  $\alpha, \beta > 0$  with  $u_\epsilon(z) \leq \alpha \exp(-\frac{\beta}{\epsilon}|z - z_\epsilon|)$  for  $\epsilon \leq \epsilon_0$ . Furthermore, with  $V_0$  and  $\Delta_0$  as in (V4),  $V(z_\epsilon) \rightarrow V_0$  and  $\Delta V(z_\epsilon) \rightarrow \Delta_0$  as  $\epsilon \rightarrow 0$ .*

For small  $\epsilon$ ,  $u_\epsilon$  resembles a “spike,” which is sharper for smaller  $\epsilon$ . The spike concentrates near a local minimum of  $V$  where  $\Delta V$  has a strict local minimum. The proof of Theorem 1.1 employs the techniques of [3], with some refinements necessary because  $V$  does not necessarily achieve a *strict* local minimum. Section 2 introduces the penalization scheme developed by Felmer and del Pino, and continues with the beginning of the proof of Theorem 1.1. These beginning arguments are taken practically verbatim from [3], but are included, since the machinery of the penalization technique is used in the remainder of the proof. The reader is invited to consult [3] for more complete proofs. Section 3 contains the completion of the proof, which is original. This part contains delicate computations involving  $\Delta V$ .

## 2 The penalization scheme

Extend  $f$  to the negative reals by defining  $f(q) = 0$  for  $q < 0$ . Let  $F$  be the primitive of  $f$ , that is,  $F(q) = \int_0^q f(t) dt$ . Define the functional  $I_\epsilon$  on  $W^{1,2}(\mathbb{R}^N)$  by

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(z)u^2) - F(u) dz.$$

$I_\epsilon$  is a  $C^1$  functional, and there is a one-to-one correspondence between positive critical points of  $I_\epsilon$  and ground states of (1.1). It is well known that  $I_\epsilon$  and similar functionals in related problems fail the Palais-Smale condition. That is, a “Palais-Smale sequence,” defined as a sequence  $(u_m)$  with  $I_\epsilon(u_m)$  convergent and  $I'_\epsilon(u_m) \rightarrow 0$  as  $m \rightarrow \infty$ , need not have a convergent subsequence. To get around this difficulty, we formulate a “penalized” problem, with a corresponding “penalized” functional satisfying the Palais-Smale condition, by altering  $f$  outside of  $\Lambda$ .

Let  $\theta$  be as in (F4). Choose  $k$  so  $k > \theta/(\theta - 2)$ . Let  $V_-$  be as in (V3) and  $a > 0$  be the value at which  $f(a)/a = V_-/k$ . Define  $\tilde{f}$  by

$$\tilde{f}(s) = \begin{cases} f(s) & s \leq a; \\ sV_-/k & s > a, \end{cases} \quad (2.1)$$

$g(\cdot, s) = \chi_\Lambda f(s) + (1 - \chi_\Lambda)\tilde{f}(s)$ , and  $G(z, \xi) = \int_0^\xi g(z, \tau) d\tau$ . Although not continuous,  $g$  is a Carathéodory function. For  $\epsilon > 0$ , define the penalized functional  $J_\epsilon$  on  $W^{1,2}(\mathbb{R}^N)$  by

$$J_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\epsilon^2|\nabla u|^2 + V(z)u^2) - G(z, u) dz. \quad (2.2)$$

A positive critical point of  $J_\epsilon$  is a weak solution of the “penalized equation”

$$-\epsilon^2\Delta u + V(z)u = g(z, u), \quad (2.3)$$

that is, a  $C^1$  function satisfying (2.3) wherever  $g$  is continuous. It is proven in [3] that  $J_\epsilon$  satisfies all the hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([2]), including the Palais-Smale condition. Therefore  $J_\epsilon$  has a critical point  $u_\epsilon$ , with the mountain pass critical level  $c(\epsilon) = J_\epsilon(u_\epsilon)$ .  $c(\epsilon)$  is defined by the following minimax: let the set of paths  $\Gamma_\epsilon = \{\gamma \in C([0, 1], W^{1,2}(\mathbb{R}^N)) \mid \gamma(0) = 0, J_\epsilon(\gamma(1)) < 0\}$ , and

$$c(\epsilon) = \inf_{\gamma \in \Gamma_\epsilon} \max_{\theta \in [0, 1]} J_\epsilon(\gamma(\theta)).$$

As shown in ([3]), because of (F4),  $c(\epsilon)$  can be characterized more simply as

$$c(\epsilon) = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} J_\epsilon(\tau u).$$

The functions  $g(z, q)$  and  $f(q)$  agree whenever  $z \in \Lambda$  or  $q < a$ . Therefore if  $u$  is a weak solution of (2.3) with  $u < a$  on  $\Lambda^C \equiv \mathbb{R}^N \setminus \Lambda$ , then  $u$  solves (1.1). Our plan is to find a positive critical point  $u_\epsilon$  of  $J_\epsilon$ , which is a weak solution of (2.3), then show that  $u_\epsilon(z) < a$  for all  $z \in \Lambda^C$ .

For  $\epsilon > 0$ , let  $u_\epsilon$  be a critical point of  $J_\epsilon$  with  $J_\epsilon(u_\epsilon) = c(\epsilon)$ . Maximum principle arguments show that  $u_\epsilon$  must be positive. Define the “limiting functional”  $I_0$  by

$$I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0u^2) - F(u) \quad (2.4)$$

and

$$c = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} I_0(\tau u). \quad (2.5)$$

The equation corresponding to (2.4) is

$$-\Delta u + V_0u = f(u) \quad (2.6)$$

We will prove Theorem 1.1 by proving the following proposition:

**Proposition 2.1** *Let  $\epsilon > 0$ . If  $u_\epsilon$  is a positive solution of (2.3) satisfying  $J_\epsilon(u_\epsilon) = c(\epsilon)$ , then*

- (i)  $\lim_{\epsilon \rightarrow 0} \max_{z \in \partial\Lambda} u_\epsilon = 0$ .
- (ii) *For all  $\epsilon$  sufficiently small,  $u_\epsilon$  has only one local maximum point in  $\Lambda$  (call it  $z_\epsilon$ ), with  $\lim_{\epsilon \rightarrow 0} V(z_\epsilon) = V_0$*
- (iii)  $\lim_{\epsilon \rightarrow 0} \Delta V(z_\epsilon) = \Delta_0$ .

**Proof of Theorem 1.1:** Assuming Proposition 2.1, there exists  $\epsilon_0 > 0$  such that for  $\epsilon < \epsilon_0$ ,  $u_\epsilon < a$  on  $\partial\Lambda$ . In [3] it is shown that if we multiply (2.3) by  $(u_\epsilon - a)_+$  and integrate by parts, it follows that  $u_\epsilon < a$  on  $\Lambda^C$ , so  $u_\epsilon$  solves (1.1). By the definition of  $a$  in (2.1), and the maximum principle,  $u_\epsilon$  has no local maxima outside of  $\Lambda$ , so  $u_\epsilon$  has exactly one local maximum point  $z_\epsilon$ , which occurs in  $\Lambda$ .

Define  $v_\epsilon$  by translating  $u_\epsilon$  from  $z_\epsilon$  to zero and dilating it by  $\epsilon$ , that is,

$$v_\epsilon(z) = u_\epsilon(z_\epsilon + \epsilon z).$$

Then  $v_\epsilon$  is a weak ( $C^1$ ) solution of the “translated and dilated” equation

$$-\Delta v_\epsilon + V(z_\epsilon + \epsilon z)v_\epsilon = g(z_\epsilon + \epsilon z, v_\epsilon).$$

Let  $\epsilon_j \rightarrow 0$ . Along a subsequence (called  $(z_{\epsilon_j})$ ),  $z_{\epsilon_j} \rightarrow \bar{z} \in \bar{\Lambda}$ , with  $V(\bar{z}) = V_0$  and  $\Delta V(\bar{z}) = \Delta_0$ .

Along a subsequence,  $v_{\epsilon_j}$  converges locally uniformly to a function  $v^0$ . Pick  $R > 0$  so  $v^0 < a$  on  $\mathbb{R}^N \setminus B_R(0)$ . For large enough  $\epsilon$ ,  $v_\epsilon < a$  on  $\partial B_R(0)$ . By the maximum principle arguments of [3], for small  $\epsilon$ ,  $v_\epsilon$  decays exponentially, uniformly in  $\epsilon$ . ◇

The proof of Proposition 2.1 will follow if we can prove the following statement.

**Proposition 2.2** *If  $\epsilon_n \rightarrow 0$  and  $(z_n) \subset \bar{\Lambda}$  with  $u_{\epsilon_n}(z_n) \geq b > 0$ , then*

- (i)  $\lim_{n \rightarrow \infty} V(z_n) = V_0$ .
- (ii)  $\lim_{n \rightarrow \infty} \Delta V(z_n) = \Delta_0$ .

It is proven in [3] that  $u_\epsilon$  has exactly one local maximum point  $z_\epsilon$  for small  $\epsilon$ . Since  $u_\epsilon$  solves (2.3), the maximum principle implies that  $u_\epsilon(z_\epsilon)$  is bounded away from zero. Thus Proposition 2.2 and (V4) give Proposition 2.1(ii)-(iii).

To prove Proposition 2.2, let  $b$  and  $(z_n)$  be as above. First we repeat the argument of [3] to show that  $V(z_n) \rightarrow V_0$ : suppose this does not happen. Then, along a subsequence,  $z_n \rightarrow \bar{z} \in \bar{\Lambda}$  with  $V(\bar{z}) > V_0$ . Define  $v_n$  by translating  $u_{\epsilon_n}$  from  $z_n$  to 0 and dilating by  $\epsilon_n$ ; that is,

$$v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z). \tag{2.7}$$

$v_n$  solves the “translated and dilated” penalized equation

$$-\Delta v_n + V(z_n + \epsilon_n z)v_n = g(z_n + \epsilon_n z, v_n) \quad (2.8)$$

on  $\mathbb{R}^N$ , with  $v_n(z) \rightarrow 0$  and  $\nabla v_n(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ . As shown in [3],  $(v_n)$  is bounded in  $W^{1,2}(\mathbb{R}^N)$ , so by elliptic estimates,  $(v_n)$  converges locally along a subsequence (also denoted  $(v_n)$ ) to  $v^0 \in W^{1,2}(\mathbb{R}^N)$ . Define  $\chi_n$  by  $\chi_n(z) = \chi_\Lambda(z_n + \epsilon_n z)$ , where  $\chi_\Lambda$  is the characteristic function of  $\Lambda$ .  $\chi_n$  converges weakly in  $L^p$  over compact sets to a function  $\chi$ , for any  $p > 1$ , with  $0 \leq \chi \leq 1$ . Define

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s)$$

Then  $v^0$  satisfies

$$-\Delta v + V(\bar{z})v = \bar{g}(z, v) \quad (2.9)$$

on  $\mathbb{R}^N$ . Define  $\bar{G}(z, s) = \int_0^s \bar{g}(z, t) dt$ . Associated with (2.9) we have the limiting functional  $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(\bar{z})u^2) - \bar{G}(z, u) dz$ .  $v^0$  is a positive critical point of  $\bar{J}$ .

Define  $J_n$  to be the “translated and dilated” penalized functional corresponding to (2.8), that is,

$$J_n(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(z_n + \epsilon_n z)u^2) - G(z_n + \epsilon_n z, u) dz.$$

Clearly  $J_n(v_n) = \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n})$ . In [3] it is proven that

$$\liminf_{n \rightarrow \infty} J_n(v_n) \geq \bar{J}(v^0). \quad (2.10)$$

Also, by letting  $w$  be a ground state for (2.6) with  $I_0(w) = \underline{c}$  (the mountain pass value for  $I_0$ , defined in (2.5) and using  $w$  as a test function for  $J_n$ , it is proven that  $\underline{c} \geq \liminf_{n \rightarrow \infty} J_n(v_n)$ ). Thus  $\bar{J}(v^0) \leq \underline{c}$ . Therefore, as shown in [3],  $V(\bar{z}) \leq V_0$ . This contradicts our assumption. Thus  $V(z_n) \rightarrow V_0$ . All the above is the same as was proven in [3]. Next, we must show that  $\Delta V(z_n) \rightarrow \Delta_0$ . That is the focus of the next section.

### 3 The effect of the Laplacian

Proving  $\Delta V(z_n) \rightarrow \Delta_0$  is a subtle and delicate problem. Making  $\epsilon_n$  approach 0 is equivalent to dilating  $V$ , which has the effect of making local minima of  $V$  behave more like global minima. This assists in finding solutions to (1.1). However, making  $\epsilon_n$  small *reduces* the effect of differences in  $\Delta V$ . For this reason, Theorem 1.1 is not only difficult to prove, but is not intuitively compelling, either.

It is known ([7]) that a “least energy solution” of (2.6), that is, a solution  $w$  with  $I_0(w) = \underline{c}$ , must be radially symmetric. We will need to exploit this fact. In order to do this, we will need to work with the maximum points of  $u_{\epsilon_n}$  instead of merely the  $(z_n)$  as given in Proposition 2.2. We need the following concentration-compactness result, which states that the sequence  $(u_{\epsilon_n})$  of “mountain-pass type solutions” of (2.3) does not “split”:

**Lemma 3.1** *If  $(z_n) \subset \bar{\Lambda}$ ,  $(y_n) \subset \mathbb{R}^N$ , and  $b > 0$  with  $u_{\epsilon_n}(z_n) > b$  and  $u_{\epsilon_n}(y_n) > b$  for all  $n$ , then  $((z_n - y_n)/\epsilon_n)$  is bounded.*

**Proof:** define  $v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z)$  as in (2.7). Suppose the lemma is false. Then, along a subsequence,  $|y_n - z_n|/\epsilon_n \rightarrow \infty$ . Let  $x_n = (y_n - z_n)/\epsilon_n$ . ( $\|v_n\|$ ) is bounded in  $W^{1,2}(\mathbb{R}^N)$  and  $|x_n| \rightarrow \infty$ , so we may pick a sequence  $(R_n) \subset \mathbb{N}$  with  $R_n \rightarrow \infty$ ,  $|x_n| - R_n \rightarrow \infty$ , and  $\|v_n\|_{W^{1,2}(B_{R_n+1}(0) \setminus B_{R_n-1}(0))} \rightarrow 0$  as  $n \rightarrow \infty$ . Define cutoff functions  $\varphi_n^{1,2} \in C^\infty(\mathbb{R}^N, [0, 1])$  satisfying  $\varphi_1 \equiv 1$  on  $B_{R_n-1}(0)$ ,  $\varphi_1 \equiv 0$  on  $B_{R_n}(0)^c$ ,  $\varphi_2 \equiv 1$  on  $B_{R_n+1}(0)^c$ ,  $\varphi_2 \equiv 0$  on  $B_{R_n}(0)$ , and  $\|\nabla \varphi_1\|_{L^\infty(\mathbb{R}^N)} < 2$ ,  $\|\nabla \varphi_2\|_{L^\infty(\mathbb{R}^N)} < 2$ . Set  $v_n^1 = \varphi_n^1 v_n$  and  $v_n^2 = \varphi_n^2 v_n$ , and  $\bar{v}_n = v_n^1 + v_n^2 = (\varphi_n^1 + \varphi_n^2)v_n$ .

Choose  $T_n > 0$  so  $J_n(T_n \bar{v}_n) = 0$ . We claim that  $T_n$  is well-defined, and bounded in  $n$ . Note that the existence of  $T_n$  must be checked for the penalized functional  $J_n$ , because of the replacement of  $F$  with  $G$ . By elliptic estimates, there exists an open set  $U \subset \mathbb{R}^N$  such that along a subsequence,  $v_n^1 > b/2$  on  $U$  and  $U \subset (\Lambda - z_n)/\epsilon_n \equiv \{z \in \mathbb{R}^N \mid z_n + \epsilon_n z \in \Lambda\}$ . Let  $a$  be as in (2.1). For  $t > 2a/b$  and  $z \in U$ ,  $t\bar{v}_n(z) > tb/2 > a$ , so  $G(z_n + \epsilon_n z, t\bar{v}_n) = F(t\bar{v}_n) > F(bt/2)$ . Therefore, for  $t > 2a/b$ ,

$$\begin{aligned} J_n(t\bar{v}_n) &= t^2 \int_{\mathbb{R}^N} \frac{1}{2} (|\nabla \bar{v}_n|^2 + V(z_n + \epsilon_n z) \bar{v}_n^2) dz - \int_{\mathbb{R}^N} G(z_n + \epsilon_n z, t\bar{v}_n) dz \\ &\leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - \int_U F(t\bar{v}_n) \\ &\leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - \lambda(U) F(tb/2), \end{aligned}$$

where  $\lambda$  indicates the Lebesgue measure. By (F4), there exists  $C > 0$  such that for  $t > 2a/b$ ,  $F(tb/2) > Ct^\theta$ . Therefore, for  $t > 2a/b$ ,

$$J_n(t\bar{v}_n) \leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - Ct^\theta. \tag{3.1}$$

Since  $(\bar{v}_n)$  is bounded in  $W^{1,2}(\mathbb{R}^N)$ , this gives the existence and boundedness of  $(T_n)$ .

Since  $J_n(T_n \bar{v}_n) = J_n(T_n v_n^1) + J_n(T_n v_n^2) = 0$ , we may pick  $i_n \in \{1, 2\}$  with  $J_n(T_n v_n^{i_n}) \leq 0$ . By (F5) and (2.1), the map  $t \mapsto J_n(tv_n^{i_n})$  increases from zero at  $t = 0$ , achieves a positive maximum, then decreases to  $-\infty$ . We will see more of this in a moment. Thus there exists a unique  $t_n \in (0, T_n)$  with  $J_n(t_n v_n^{i_n}) = \max_{t>0} J_n(tv_n^{i_n})$ . We claim that  $t_n$  and  $T_n - t_n$  are both bounded away from zero for large  $n$ : by  $(f_1) - (f_4)$  and (2.1),  $J_n(w) \geq \frac{1}{\theta} \min(1, V_-) \|w\|_{W^{1,2}(\mathbb{R}^N)}^2 - o(\|w\|_{W^{1,2}(\mathbb{R}^N)}^2)$  uniformly in  $n$ , so  $\max_{t>0} J_n(tv_n^{i_n})$  is bounded away from zero, uniformly in  $n$ . It is easy to show that  $J_n$  is Lipschitz on bounded subsets of  $W^{1,2}(\mathbb{R}^N)$ , uniformly in  $n$ . Since  $(T_n)$  is bounded, this implies that  $t_n$  and  $T_n - t_n$  are both bounded away from zero for large  $n$ .

By definition of  $v_n$  as a ‘‘mountain-pass type critical point’’ of  $J_n$ , we have

$$\max_{t>0} J_n(tv_n^{i_n}) \geq \max_{t>0} J_n(tv_n).$$

Using the facts that  $\|v_n - \bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)} \rightarrow 0$  as  $n \rightarrow \infty$ , and  $(T_n)$  is bounded, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_n(t_n v_n^{i_n}) &= \liminf_{n \rightarrow \infty} \max_{t>0} J_n(t v_n^{i_n}) \\ &\geq \liminf_{n \rightarrow \infty} \max_{t>0} J_n(t v_n) \\ &= \liminf_{n \rightarrow \infty} \max_{t>0} J_n(t \bar{v}_n) \tag{3.2} \\ &= \liminf_{n \rightarrow \infty} J_n(t_n \bar{v}_n) \\ &= \liminf_{n \rightarrow \infty} (J_n(t_n v_n^{i_n}) + J_n(t_n v_n^{3-i_n})) \\ &\geq \liminf_{n \rightarrow \infty} J_n(t_n v_n^{i_n}) + \liminf_{n \rightarrow \infty} J_n(t_n v_n^{3-i_n}). \end{aligned}$$

Now  $J_n(T_n v_n^{3-i_n}) = -J_n(T_n v_n^{i_n}) \geq 0$  and  $t_n < T_n$ , so  $J_n(t_n v_n^{3-i_n}) \geq 0$ . By (3.2),  $\liminf_{n \rightarrow \infty} J_n(t_n v_n^{3-i_n}) \leq 0$ . Therefore  $J_n(t_n v_n^{3-i_n}) \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $J_n(w) \geq \frac{1}{\theta} \min(1, V_-) \|w\|_{W^{1,2}(\mathbb{R}^N)}^2 - o(\|w\|_{W^{1,2}(\mathbb{R}^N)}^2)$  uniformly in  $n$ , there exists  $d \in (0, \liminf_{n \rightarrow \infty} t_n)$  such that  $\liminf_{n \rightarrow \infty} J_n(d v_n^{3-i_n}) > 0$ . Since  $d < t_n$  and  $J_n(d v_n^{3-i_n}) > J_n(t_n v_n^{3-i_n})$  for large  $n$ , the map  $t \mapsto J_n(t v_n^{3-i_n})$  achieves a maximum at some  $t'_n \in (0, t_n)$ , and that maximum is bounded away from zero.

Summarizing the important facts about the mapping  $t \mapsto J_n(t v_n^{3-i_n})$ , we have shown that there exists  $\rho > 0$  such that for large  $n$ ,

- (i)  $0 < t'_n < t_n < T_n$
- (ii)  $(T_n)$  is bounded.
- (iii)  $(T_n - t_n)$  is bounded away from zero.
- (iv)  $J_n(t'_n v_n^{3-i_n}) > \rho > 0$
- (v)  $J_n(t_n v_n^{3-i_n}) \rightarrow 0$
- (vi)  $J_n(T_n v_n^{3-i_n}) \geq 0$

From (i)-(vi) it is apparent that at some  $t_n^* > t'_n$ , the mapping  $t \mapsto J_n(t v_n^{3-i_n})$  is at once decreasing and concave upward. But this is impossible: let  $n \in \mathbb{N}$  and  $w \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}$ . Define  $\psi(t) = J_n(tw)$  for  $t > 0$ . Then

$$\begin{aligned} \psi'(t) &= t \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2 dz - \int_{\mathbb{R}^N} g(z_n + \epsilon_n z, tw) w dz \\ &= t \left[ \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2 dz - \int_{\{w \neq 0\}} \frac{g(z_n + \epsilon_n z, tw)}{tw} w^2 dz \right]. \end{aligned}$$

By (F5) and (2.1),  $t \mapsto g(z_n + \epsilon_n z, tw)/(tw)$  is nondecreasing, so if  $\psi'(t)$  ever becomes negative,  $\psi'$  is increasing for all time  $t$  after that, and the graph of  $\psi$  is concave down. Therefore the behavior of  $J_n(t v_n^{3-i_n})$  as described in (i)-(vi) is impossible, and Lemma 3.1 is proven.  $\diamond$



As mentioned before, it will be advantageous to work with the maxima of  $(u_{\epsilon_n})$ . Choose  $(y_n) \subset \mathbb{R}^N$  with

$$u_{\epsilon_n}(y_n) = \max_{\mathbb{R}^N} u_{\epsilon_n}.$$

We will prove

$$\Delta V(y_n) \rightarrow \Delta_0. \tag{3.3}$$

By Lemma 3.0,  $((y_n - z_n)/\epsilon_n)$  is bounded, so  $y_n - z_n \rightarrow 0$ . Thus (3.3) gives Proposition 2.2(ii), completing the proof of Theorem 1.1.  $\diamond$

Along a subsequence,  $y_n \rightarrow \bar{y} \in \bar{\Lambda}$ . By Proposition 2.2(i),  $V(\bar{y}) = V_0$ . Since is not apparent that  $\bar{y} \in \Lambda$ , we must proceed carefully. We will redefine the  $v_n$ 's like in (2.7), by translating  $u_{\epsilon_n}$  to 0 and dilating it. That is,

$$v_n(z) = u_{\epsilon_n}(y_n + \epsilon_n z). \tag{3.4}$$

Then  $v_n$  is a positive weak solution, vanishing at infinity, of the “penalized, dilated, and translated” PDE

$$-\Delta v + V(y_n + \epsilon_n z)v = g(y_n + \epsilon_n z, v).$$

Like before,  $(v_n)$  converges locally uniformly to a function  $v_0$ . We claim that  $v_0$  is actually a ground state maximizing at 0 of the autonomous limiting equation (2.6). Proof: As before, define  $\chi_n$  by  $\chi_n(z) = \chi(y_n + \epsilon z)$ . As before, along a subsequence,  $\chi_n$  converges weakly in  $L^p$ , for any  $p > 1$ , on compact subsets of  $\mathbb{R}^N$  to a function  $\chi$  with  $0 \leq \chi \leq 1$ . Define  $\bar{g}$  by

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s).$$

By the argument of Proposition 2.2, taken from [3],  $(v_n)$  converges locally along a subsequence to  $v_0$ , a ground state of  $-\Delta v + V_0 v = \bar{g}(z, v)$ . The functional corresponding to this equation is  $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - \bar{G}(z, u) dz$ , where  $\bar{G}(z, s) = \int_0^s \bar{g}(z, t) dt$ . As before, in (2.10),  $\underline{c} \geq \liminf_{n \rightarrow \infty} J_n(v_n) \geq \bar{J}(v_0)$ , where  $\underline{c}$  is from (2.5).  $\bar{J} \geq I_0$ , where  $I_0$  is the “autonomous” limiting functional from (2.4), so

$$\underline{c} \leq \max_{t>0} I_0(tv_0) \leq \max_{t>0} \bar{J}(tv_0) \leq \underline{c},$$

and  $v_0$  is actually a ground state of (2.6).  $\diamond$

Not only does  $(v_n)$  converge locally to  $v_0$ , but it satisfies the following lemma.

**Lemma 3.2** *With  $(v_n)$  as in (3.4), for any subsequence of  $(v_n)$  there is a radially symmetric ground state  $v_0$  of (2.6) such that  $v_n \rightarrow v_0$  uniformly along a subsequence and the  $v_n$ 's decay exponentially, uniformly in  $n$ .*

**Proof:** If one establishes uniform convergence, the uniform exponential decay follows readily, using a standard maximum principle argument found in [3]. Suppose the convergence is not uniform. Then there exist a subsequence of  $(v_n)$  (denoted  $(v_n)$ ) and a sequence  $(x_n) \subset \mathbb{R}^N$  with  $|x_n| \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} v_n(x_n) > 0$ . Let  $d > 0$  with  $d < v_0(0)$  and  $d < \lim_{n \rightarrow \infty} v_n(x_n)$ . For large  $n$ ,  $d < v_n(0) = u_{\epsilon_n}(z_n)$  and  $d < v_n(x_n) = u_{\epsilon_n}(z_n + \epsilon_n x_n)$ . Letting  $w_n = z_n + \epsilon_n x_n$ , we obtain  $((w_n - z_n)/\epsilon_n) = (x_n)$ , which is unbounded, violating Lemma 3.1.

To show  $\Delta V(y_n) \rightarrow \Delta_0$ , we again argue indirectly. Suppose otherwise. Then, along a subsequence,  $y_n \rightarrow \bar{y} \in \bar{\Lambda}$  with

$$\Delta V(\bar{y}) > \Delta_0. \quad (3.5)$$

For  $x \in \mathbb{R}^N$ , define the translation operator  $\tau_x$  by  $\tau_x u(z) = u(z - x)$ , that is,  $\tau_x u$  is  $u$  translated by  $x$ . Assume for convenience, and without loss of generality, that

$$0 \in \Lambda, \quad V(0) = V_0, \quad \text{and} \quad \Delta V(0) = \Delta_0.$$

We will prove that for large  $n$ ,

$$\sup_{t>0} J_{\epsilon_n}(t\tau_{-y_n/\epsilon_n} u_{\epsilon_n}) < J_{\epsilon_n}(u_{\epsilon_n}) = \sup_{t>0} J_{\epsilon_n}(tu_{\epsilon_n}), \quad (3.6)$$

recalling the definition of  $J_\epsilon$  in (2.2), and how  $v_n$  is defined from  $u_{\epsilon_n}$  in (3.4). That is, translating  $tu_{\epsilon_n}$  back to the origin reduces the value of  $J_{\epsilon_n}(tv_n)$  because  $V$  has lesser concavity at the origin. This occurs even though shrinking  $\epsilon$  reduces the difference in concavity. (3.6) contradicts the definition of  $u_{\epsilon_n}$ .

Pick  $T > 1$  large enough so that for large  $n$ ,  $J_n(Tv_n) = \epsilon_n^{-N} J_{\epsilon_n}(Tu_{\epsilon_n}) < 0$ . This is possible by the argument of (3.1). Now (3.6) is equivalent to

$$\sup_{0 \leq t \leq T} J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) < \sup_{0 \leq t \leq T} J_{\epsilon_n}(tu_{\epsilon_n}).$$

To prove the above, it will suffice to prove the stronger fact that for large  $n$ , for all  $t \in (0, T)$ ,

$$J_{\epsilon_n}(tu_{\epsilon_n}) > J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}).$$

Now, along a subsequence,  $v_n \rightarrow v_0$  uniformly, so by the definition of  $v_n$  as a dilation of  $\tau_{-y_n} u_{\epsilon_n}$  ((3.4)),  $u_{\epsilon_n} \rightarrow 0$  uniformly on  $\mathbb{R}^N \setminus \Lambda$  as  $n \rightarrow \infty$ . Thus for large  $n$  and  $0 \leq t \leq T$ , the definition of  $G$  gives  $G(z, t\tau_{-y_n} u_{\epsilon_n}(z)) = F(t\tau_{-y_n} u_{\epsilon_n}(z))$  for all  $z \in \mathbb{R}^N$ , so

$$\begin{aligned} & J_{\epsilon_n}(tu_{\epsilon_n}) - J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) \\ &= \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla u_{\epsilon_n}(z)|^2 + V(z)u_{\epsilon_n}(z)^2) - G(z, tu_{\epsilon_n}(z)) \, dz \\ &\quad - \left[ \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla \tau_{-y_n} u_{\epsilon_n}(z)|^2 + V(z)\tau_{-y_n} u_{\epsilon_n}(z)^2) - F(t\tau_{-y_n} u_{\epsilon_n}(z)) \, dz \right] \\ &\geq \frac{1}{2} t^2 \int_{\mathbb{R}^N} V(z)(u_{\epsilon_n}(z)^2 - u_{\epsilon_n}(z + y_n)^2) \, dz \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^N} F(tu_{\epsilon_n}(z + y_n) - F(tu_{\epsilon_n}(z)) dz \\
 = & \frac{1}{2} t^2 \int_{\mathbb{R}^N} (V(z + y_n) - V(z)) u_{\epsilon_n}(z + y_n)^2 dz \\
 = & \frac{1}{2} t^2 \epsilon_n^N \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z)) u_{\epsilon_n}(\epsilon_n z + y_n)^2 dz \\
 = & \frac{1}{2} t^2 \epsilon_n^N \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z)) v_n(z)^2 dz.
 \end{aligned}$$

For  $n = 1, 2, \dots$ , define  $h_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h_n(t) = \int_{\mathbb{R}^N} (V(y_n + tz) - V(tz)) v_n^2 dz.$$

Since  $h_n(\epsilon_n) = \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z)) v_n^2 dz$ , we must prove that for large  $n$ ,

$$h_n(\epsilon_n) > 0. \tag{3.7}$$

Assume without loss of generality that  $\Lambda$  was chosen so that there exists  $\rho > 0$  with

$$\inf_{N_\rho(\Lambda)} V = V_0, \tag{3.8}$$

where  $N_\rho(\Lambda) = \{x \in \mathbb{R}^N \mid \exists y \in \Lambda \text{ with } |y - x| < \rho\}$ . We will prove the following facts about  $h_n$ :

**Lemma 3.3** *For some  $\beta > 0$ , for large  $n$ ,*

- (i)  $h_n \in C^2(\mathbb{R}^+, \mathbb{R})$
- (ii)  $h_n(0) \geq 0$
- (iii)  $|h'_n(0)|^2 \leq o(1)h_n(0)$
- (iv)  $h''_n(0) > \beta$
- (v)  $h''_n$  is locally Lipschitz on  $\mathbb{R}^+$ , uniformly in  $n$ .

Here  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ . Before proving Lemma 3.3, let us prove how it gives (3.7). By (iv)-(v), there exists  $d > 0$  such that for large  $n$  and  $0 \leq t \leq d$ ,  $h''_n(t) > \beta/2$ . For  $t \in [0, d]$ , a Taylor's series expansion shows that for large  $n$ ,

$$h_n(t) \geq h_n(0) + h'_n(0)t + \frac{\beta}{4}t^2 \equiv l_n(t). \tag{3.9}$$

If  $h_n(0) = 0$ , then by Lemma 3.3(iii),  $h'_n(0) = 0$ , so (3.9) implies that  $h_n(t) > 0$  for all  $t \in (0, d)$ , giving (3.7) if  $n$  is large enough that  $\epsilon_n < d$ . If  $h_n(0) > 0$ , then by elementary calculus,  $l_n$  attains a minimum value at  $t = -2h'_n(0)/\beta$ , and the minimum value is

$$\min_{\mathbb{R}} l_n = l_n(-2h'_n(0)/\beta) = h_n(0) - h'_n(0)^2/\beta \geq (1 - o(1))h_n(0),$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ . For large  $n$ , if  $h_n(0) > 0$  then  $l_n(t) > 0$  for all  $t \in \mathbb{R}$ , so  $h_n(t) > 0$  for all  $t \in (0, d)$  for large  $n$ , implying (3.7) if  $n$  is large enough so that  $\epsilon_n < d$ .

**Proof of Lemma 3.3** Statement (ii) is trivial, since  $h_n(0) = (V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2$ , and since  $z_n \in \bar{\Lambda}$  and  $y_n - z_n \rightarrow 0$ , (3.8) implies  $V(y_n) \geq V_0$  for large  $n$ . (i) and (v) follow from Leibniz's Rule,  $(V_1) - (V_2)$ , and the fact that the  $v_n$ 's decay exponentially, uniformly in  $n$ . For  $j = 1, 2$ ,

$$h_n^{(j)}(t) = \int_{\mathbb{R}^N} \sum_{|\alpha|=j} (D^\alpha V(y_n + tz) - D^\alpha V(tz)) z^\alpha v_n(z)^2 dz.$$

Since (V2) holds,  $v_n$  decays exponentially, uniformly in  $n$ ,  $y_n \rightarrow \bar{y}$ , and  $v_0$  is radially symmetric, we have

$$\begin{aligned} h_n''(0) &= \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(y_n) - D^\alpha V(0)) z^\alpha v_n(z)^2 dz \\ &\rightarrow \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(\bar{y}) - D^\alpha V(0)) z^\alpha v_0(z)^2 dz \\ &= \int_{\mathbb{R}^N} \sum_{i=1}^N (D^{ii} V(\bar{y}) - D^{ii} V(0)) z_i^2 v_0(z)^2 dz \\ &= \int_{\mathbb{R}^N} \sum_{i=1}^N (D^{ii} V(\bar{y}) - D^{ii} V(0)) \frac{1}{N} |z|^2 v_0(z)^2 dz \\ &= \frac{1}{N} (\Delta V(\bar{y}) - \Delta V(0)) \int_{\mathbb{R}^N} |z|^2 v_0(z)^2 dz > 0 \end{aligned}$$

by assumption (3.5). Since Lemma 3.3(v) holds, we have Lemma 3.3(iv).

To prove Lemma 3.3(iii), we will need the following calculus lemma:

**Lemma 3.4** *Let  $U \subset \mathbb{R}^N$  and  $r > 0$ . Let  $V \in C^2(N_r(U), \mathbb{R})$  with  $\inf_{N_r(U)} V \equiv V_0 > -\infty$ ,  $|\nabla V|$  bounded on  $N_r(U)$ , and  $D^2V$  Lipschitz on  $N_r(U)$ . Then there exists  $C > 0$  with*

$$|\nabla V(z)|^2 \leq C(V(z) - V_0) \tag{3.10}$$

for all  $z \in U$ .

**Proof:** let  $B > 0$  with  $|D^2V(z)\xi \cdot \xi| \leq B$  for all  $\xi \in \mathbb{R}^N$  with  $|\xi| = 1$ . Also let  $B$  be big enough so

$$B > |\nabla V(z)|/r$$

for all  $z \in U$ . Pick  $z \in U$ . If  $|\nabla V(z)| = 0$ , then (3.10) is obvious. Otherwise, let  $d = |\nabla V(z)|/B < r$ . Define  $\varphi(t) = V(z - t\nabla V(z)/|\nabla V(z)|)$  for  $t \in [0, d]$ .  $\varphi$  is  $C^2$ ,  $\varphi(0) = V(z)$ , and  $\varphi'(0) = -|\nabla V(z)|$ . By choice of  $B$  and the fact that  $B_d(z) \subset N_r(U)$ ,  $|\varphi''(t)| \leq B$  for all  $t \in [0, d]$ . Taylor's theorem gives

$$\varphi(d) - \varphi(0) = \varphi'(0)d + \varphi''(\xi) \frac{d^2}{2} \leq -|\nabla V(z)|d + Bd^2/2 = -\frac{|\nabla V(z)|^2}{2B}.$$

Also  $\varphi(d) \geq V_0$  because  $B_d(z) \subset N_r(U)$ . Therefore,

$$\frac{|\nabla V(z)|^2}{2B} \leq \varphi(0) - \varphi(d) \leq V(z) - V_0.$$

Lemma 3.4 is proven.  $\diamond$

To prove Lemma 3.3(iii), first note that, by the radial symmetry of  $v_0$ , the uniform exponential decay of  $v_n$ , and the uniform convergence  $v_n \rightarrow v_0$ ,

$$\begin{aligned} |h'_n(0)| &= |(\nabla V(y_n) - \nabla V(0)) \cdot \int_{\mathbb{R}^N} z v_n^2 dz| \\ &= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_n^2 dz| \\ &= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_0^2 dz + \nabla V(y_n) \cdot \int_{\mathbb{R}^N} z (v_n^2 - v_0^2) dz| \\ &= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z (v_n^2 - v_0^2) dz| \\ &\leq |\nabla V(y_n)| \left| \int_{\mathbb{R}^N} z (v_n^2 - v_0^2) dz \right| \\ &\leq o(1) |\nabla V(y_n)|, \end{aligned}$$

so Lemma 3.4 implies

$$\begin{aligned} |h'_n(0)|^2 &\leq o(1) |\nabla V(y_n)|^2 \leq o(1) (V(y_n) - V_0) \\ &\leq o(1) (V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2 \\ &= o(1) h_n(0), \end{aligned}$$

since  $\int_{\mathbb{R}^N} v_n^2$  is bounded away from zero. Lemma 3.3(iii) is proven. Thence follow (3.7), (3.3), Proposition 2.2, and Theorem 1.1.

**Remarks:** Besides the results cited in the introduction, many important results for equations of type (1.1) have been found recently. For instance, the work in [3]-[5] suggests that Theorem 1.1 could be strengthened by working on a smaller domain than  $\mathbb{R}^N$ , or by weakening the hypotheses on  $V$ . It is natural to try to extend Theorem 1.1 to cases where  $V$  is not  $C^2$ , or to the case where the second derivatives of  $V$  do not provide a condition like (V4), but higher-order derivatives do.

## References

- [1] Ambrosetti, A., Badiale, M., Cingolani, S.: Semiclassical states of nonlinear Schrödinger equations. Arch. Rat. Mech. Anal. (1997) **140** no. 3, 285-300.
- [2] Ambrosetti, A., Rabinowitz, P.: Dual variational methods in critical point theory and applications. Journal of Functional Analysis (1973) **14**, 349-381.

- [3] Felmer, P., del Pino, M.: Local Mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. and PDE* (1996) **4**, 121-137.
- [4] Felmer, P., del Pino, M.: Semi-classical states for nonlinear Schrödinger equations. *J. Funct. Anal.* (1997) **149** no. 1, 245-265.
- [5] Felmer, P., del Pino, M.: Multi-peak bound states for nonlinear Schrödinger equations. *Ann. Inst. Henri Poincaré, Analyse non linéaire* (1998) **15** no. 2, 127-149.
- [6] Floer, A., Weinstein, A.: Nonspreading Wave Packets for the Cubic Schrödinger Equation with a Bounded Potential, *J. Funct. Anal.* (1986) **69**, 397-408.
- [7] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in  $\mathbb{R}^n$ . *Math. Anal. and Applications, Part A, Advances in Math. Suppl. Studies 7A* (ed. L Nachbin), Academic Press (1981), 369-402.
- [8] Li, Y.-Y.: On a singularly perturbed elliptic equation. *Advances in Differential Equations* (1997) **2**, 955-980.
- [9] Lu, G., Wei, J.: On Nonlinear Schrödinger Equations with Totally Degenerate Potentials. *C. R. Acad. Sci. Paris Série I Math.* (1998) **326** no. 6, 691-696.
- [10] Oh, Y.J.: Existence of semi-classical bound states of nonlinear Schrödinger equations with potential on the class  $(V_a)$ , *Comm. Partial Differ. Eq.* (1988) **13**, 1499-1519.
- [11] Oh, Y.J.: Corrections to Existence of semi-classical bound states of nonlinear Schrödinger equations with potential on the class  $(V_a)$ , *Comm. Partial Differ. Eq.* (1989) **14**, 833-834.
- [12] Oh, Y.J.: On positive multi-lump bound states nonlinear Schrödinger equations under multiple well potential, *Comm. Math. Phys.* (1990) **131**, 223-253.
- [13] Rabinowitz, P. H.: On a class of nonlinear Schrödinger equations. *Z. angew Math. Phys.* (1992) **43**, 270-291.
- [14] Spradlin, G.: A Singularly Perturbed Elliptic Partial Differential Equation with an Almost Periodic Term. *Calc. Var. and PDE* (1999) **9**, 207-232.
- [15] Wang, X.: On Concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.* (1993) **153**, 229-244.

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