An Almost Periodic Function of Several Variables with no Local Minimum

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1. Introduction

An almost periodic function, defined in a moment, is a generalization of a periodic function. Like periodic functions, almost periodic functions can be defined on $\mathbf{R}$, on $\mathbf{R}^n$, even on more general topological groups ([C]). Many properties of almost periodic functions on the real line are well known. The properties of almost periodic functions on other domains appear to be less well known. This paper proves that these properties may be very different.

These differences may have implications for the field of differential equations. Recently, Serra, Tarallo and Terracini ([STT]) studied an ordinary differential equation containing an almost periodic term, finding a solution homoclinic to zero. Efforts to solve the problem generalized to a PDE have failed. The proof in [STT] used topological properties of the real line not shared by $\mathbf{R}^n$ ($n \geq 2$). Different approaches, attempted by the author and others, might have succeeded had it been true that any almost periodic function on $\mathbf{R}^n$ has a local minimum. This paper shows, by exhibiting a counterexample, that this plausible, intuitive property does not hold.

Let us define an almost periodic function on $\mathbf{R}^n$. This definition is the natural generalization of (Bohr) almost periodic and is found in [Be]. First, a set $\mathcal{A} \subset \mathbf{R}^n$ is *relatively dense* if there exists $L > 0$ such that for every $x \in \mathbf{R}^n$, there exists $y \in \mathcal{A}$ with $|x - y| < L$. Next, for $\epsilon > 0$, $\vec{v} \in \mathbf{R}^n$, and $h : \mathbf{R}^n \to \mathbf{R}$, we say $\vec{v}$ is an ϵ-*almost period* of h if for all $x \in \mathbf{R}^n$, $|h(x + \vec{v}) - h(x)| < \epsilon$. Finally, h is defined to be *almost periodic* if for every $\epsilon > 0$, there exists a relatively dense set $A \equiv A(\epsilon)$ such that for all $a \in A$, a is an ϵ-almost period of h.

We will prove the following:

Theorem 1.0 *There exists a function $G \in C^\infty(\mathbf{R}^2, \mathbf{R})$ that is almost periodic and has no local minimum.*

Of course, $y \in \mathbf{R}^2$ is defined to be a *local minimum* of G if there exists $\epsilon > 0$ such that $G(x) \geq G(y)$ for all $x \in B_\epsilon(y)$, that is, for all $x \in \mathbf{R}^2$ with $|x - y| < \epsilon$. By contrast, it is easy to show that any almost periodic function on the reals must have infinitely many local minima. Note: Theorem 1.0 implies that for *any* $n \geq 2$, there is an almost periodic function on $\mathbf{R}^n$ with no local minimum: let G be as above and define

$$\bar{G}(x) = \bar{G}(x_1, x_2, \ldots, x_n) = G(x_1, x_2). \tag{1.1}$$

Then $\bar{G} \in C^\infty(\mathbf{R}^n, \mathbf{R})$ has no local minima, because G has none.

This paper is organized as follows: in Section 2, to make reasoning simpler, we formulate a discrete version of the problem. That is, we define notions of almost periodicity and local minimum for functions $\mathbf{Z}^2 \to \mathbf{R}$, then construct a function $g : \mathbf{Z}^2 \to \mathbf{R}$ that satisfies an analogue of Theorem 1.0. In Section 3, we extend g to $\mathbf{R}^2$ so that it is infinitely differentiable, almost periodic, and has no local minima.

2. The Discretized Problem

Let us generalize "almost periodic" to functions on $\mathbf{Z}^2$ in the most obvious way. If $f : \mathbf{Z}^2 \to \mathbf{R}$, $\epsilon > 0$, and $a \in \mathbf{Z}^2$, say that a is an ϵ-*almost period* of f if $|f(x+a) - f(x)| < \epsilon$ for all $x \in \mathbf{Z}^2$. Define f to be almost periodic if, for every $\epsilon > 0$, there exists a set of ϵ-almost periods of f that is relatively dense in $\mathbf{R}^2$. Define the metric d on $\mathbf{Z}^2$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|. \tag{2.0}$$

We will construct $g : \mathbf{Z}^2 \to \mathbf{R}$ satisfying:

(2.1) $\quad (i)$ g is almost periodic,

and for all $x \in \mathbf{Z}^2$,

(ii) $g(y) \neq g(x)$ for all $y \in \mathbf{Z}^2$ with $d(x, y) = 1$, and

(iii) There exists $y \in \mathbf{Z}^2$ with $d(y, x) = 1$ and $g(y) < g(x)$.

Define the basis unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let us say a function f from $\mathbf{Z}^2$ to $\mathbf{R}$ is "a-periodic" if $f(x) = f(x + a)$ for all $x \in \mathbf{Z}^2$. For $l = 1, 2, \ldots$, we will construct $g_l : \mathbf{Z}^2 \to \mathbf{R}$ satisfying

(2.2) $\quad (i)$ $0 \leq g_l(x) \leq 3^{l+1}$ for all $x \in \mathbf{Z}^2$, and

(ii) g_l is $2 \cdot 3^l e_1$- and $2 \cdot 3^l e_2$- periodic.

Then define

$$(2.3) \qquad g = \sum_{l=1}^{\infty} \left(\frac{1}{10}\right)^{l^2} g_l.$$

g is well-defined and almost periodic: the above series converges because for any $x \in \mathbf{Z}^2$,

$$(2.4) \qquad 0 \leq g(x) \leq \sum_{l=1}^{\infty} \left(\frac{1}{10}\right)^{l^2} 3^{l+1} \leq 3 \sum_{l=1}^{\infty} \left(\frac{3}{10}\right)^{l} < \infty.$$

g is almost periodic, since a convergent series of periodic functions is almost periodic.

Now we define g_l more precisely. For $l = 1, 2, \ldots$, define g_l by

$$(2.8) \qquad g_l(x) = \begin{cases} 2 \cdot 3^l - |x_1| - |x_2|; & |x_1| \leq 3^l - 1 \text{ and } |x_2| \leq 3^l - 1, \\ 0; & |x_1| = 3^l \text{ or } |x_2| = 3^l, \\ g_l(x + 2 \cdot 3^l y) = g_l(x) & \text{for all } x, y \in \mathbf{Z}^2. \end{cases}$$

This definition is consistent. Figure 2.9 shows g_1 (with $g_1(0,0) = 6$).

$$\begin{array}{ccccccccccc} & & & & & \vdots & & & & & \\ & 2 & 0 & 2 & 3 & 4 & 3 & 2 & 0 & 2 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 2 & 0 & 2 & 3 & 4 & 3 & 2 & 0 & 2 & \\ & 3 & 0 & 3 & 4 & 5 & 4 & 3 & 0 & 3 & \\ \cdots & 4 & 0 & 4 & 5 & 6 & 5 & 4 & 0 & 4 & \cdots \\ & 3 & 0 & 3 & 4 & 5 & 4 & 3 & 0 & 3 & \\ & 2 & 0 & 2 & 3 & 4 & 3 & 2 & 0 & 2 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 2 & 0 & 2 & 3 & 4 & 3 & 2 & 0 & 2 & \\ & & & & & \vdots & & & & & \end{array}$$

Figure 2.9 g_1 for $|x_1|, |x_2| \leq 4$.

g_l can be thought of as a "bump" supported in the square $\{|x_1| \leq 3^l - 1, |x_2| \leq 3^l - 1\}$ and repeated periodically. g_l obviously satisfies (2.2)(i)-(ii). g_l also satisfies the following properties (some of which can be checked for $l = 1$ using Figure 2.9):

(2.10) (i) $g_l(x) \in \mathbf{Z}$ for all $x \in \mathbf{Z}^2$.

(ii) $|x_1| \leq 3^l - 1$ and $|x_2| \leq 3^l - 1 \Rightarrow g_l(x) > 0$.

(iii) $g_l(x) = 0 \Rightarrow g_1(x) = g_2(x) = \ldots = g_{l-1}(x) = 0$.

(iv) $g_l(x) = 0 \Rightarrow g_l(x - e_1) = 0 = g_l(x + e_1)$ or $g_l(x - e_2) = 0 = g_l(x + e_2)$.

(v) $g_l(x) > 0 \Rightarrow g_l(y) \neq g_l(x)$ for all $y \in \mathbf{Z}^2$ with $d(y, x) = 1$.

(vi) $g_l(x) > 0 \Rightarrow \{g_l(x - e_1) < g_l(x)$ or $g_l(x + e_1) < g_l(x)\}$ and

$\{g_l(x - e_2) < g_l(x)$ or $g_l(x + e_2) < g_l(x)\}$.

Proof: (2.10)(i) and (ii) are obvious. To prove (iii)-(iv), define S_l to be the zero set of g_l, that is,

$$S_l \equiv \{z \in \mathbf{Z}^2 \mid g_l(z) = 0\} = \tag{2.11}$$
$$= \{z \in \mathbf{Z}^2 \mid z = (3^l n_1, 3^l n_2),\ n_1 \text{ or } n_2 \text{ is an odd integer.}\}$$

$S_1 \supset S_2 \supset S_3 \supset \ldots$, proving (iii). (iv) is also obvious from the form of S_l. To prove (v), let $x \in \mathbf{Z}^2$ with $g_l(x) > 0$. Since g_l is $2 \cdot 3^l e_1$- and $2 \cdot 3^l e_2$- periodic, and $g_l(x) > 0$, assume without loss of generality that $|x_1|$ and $|x_2|$ are less than or equal to $3^l - 1$. Let $y \in \mathbf{Z}^2$ with $d(x, y) = 1$. If $x_1 = y_1$ then $|x_2 - y_2| = 1$. If $|y_2| = 3^l$ also, then $g_l(y) = 0 \neq g_l(x)$. So assume $|y_2| \leq 3^l - 1$. Then

$$g_l(x) - g_l(y) = |y_2| - |x_2| \neq 0, \tag{2.12}$$

because x_2 and y_2 are integers that differ by 1. If $x_1 \neq y_1$ then $|x_1 - y_1| = 1$ and $x_2 = y_2$. If $|y_1| = 3^l$ also, then $g_l(y) = 0 \neq g_l(x)$, so assume $|y_1| \leq 3^l - 1$. Then

$$g_l(x) - g_l(y) = |y_1| - |x_1| \neq 0. \tag{2.13}$$

(v) is proven.

To prove (2.10)(vi), let $x \in \mathbf{Z}^2$ with $g_l(x) > 0$. Since g_l is $2 \cdot 3^l e_1$- and $2 \cdot 3^l e_2$- periodic and symmetric with respect to the x_1- and x_2-axes, and $g_l(x) > 0$, assume without loss of generality that x_1 and x_2 are between 0 and $3^l - 1$, inclusive . Then $g_l(x + e_1)$ and $g_l(x + e_2)$ are both less than $g_l(x)$, proving (vi): If $x_1 = 3^l - 1$, then $g_l(x + e_1) = 0 < g_l(x)$. If $x_1 < 3^l - 1$, then $g_l(x + e_1) = g_l(x) - 1$ by the definition of g_l. The reasoning is the same for $g_l(x + e_2)$.

Now we can prove that g has properties (2.1)(ii)-(iii). Let $x \in \mathbf{Z}^2$ and $y \in \mathbf{Z}^2$ with $d(y, x) = 1$. By (2.10)(ii) and (2.10)(v), for large enough l, $g_l(y) \neq g_l(x)$. Let $N \geq 1$ with $g_N(y) \neq g_N(x)$ and $g_l(y) = g_l(x)$ for all $l = 1, 2, \ldots, N - 1$. Then

$$|g(x) - g(y)| = |\sum_{l=1}^{\infty} (\frac{1}{10})^{l^2} (g_l(x) - g_l(y))| = |\sum_{l=N}^{\infty} (\frac{1}{10})^{l^2} (g_l(x) - g_l(y))| \geq \tag{2.14}$$
$$\geq (\frac{1}{10})^{N^2} - \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2} |g_l(x) - g_l(y)| \geq (\frac{1}{10})^{N^2} - \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2} 3^{l+1} =$$
$$= (\frac{1}{10})^{N^2} - (\frac{1}{10})^{N^2} \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2 - N^2} 3^{l+1}.$$

For $l \geq N+1$,

$$l^2 - N^2 = (l+N)(l-N) \geq l+N \geq l+1. \tag{2.15}$$

Therefore

$$\sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2-N^2} 3^{l+1} \leq \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l+1} 3^{l+1} = \frac{10}{7}(\frac{3}{10})^{N+2} \leq \frac{270}{7000} < \frac{1}{10}. \tag{2.16}$$

Therefore by (2.14)-(2.16),

$$|g(x) - g(y)| > (\frac{1}{10})^{N^2} (1 - \frac{1}{10}) > 0, \tag{2.17}$$

proving (2.1)(ii).

To prove (2.1)(iii), let $x \in \mathbf{Z}^2$. By (2.10)(ii)-(iii), there exists $N \geq 1$ with $g_N(x) > 0$ and $g_l(x) = 0$ for all $l = 1, 2, \dots, N-1$. By (2.10)(iv), $g_l(x+\hat{e}) = g_l(x-\hat{e}) = 0$ for all $l = 1, 2, \dots, N-1$, where $\hat{e} = e_1$ or e_2. Assume for simplicity that $\hat{e} = e_1$. By (2.10)(vi), $g_N(x+e_1) < g_N(x)$ or $g_N(x-e_1) < g_N(x)$. Also assume for simplicity that $g_N(x+e_1) < g_N(x)$. Then

$$\begin{aligned} g(x)-g(x+e_1) &= \sum_{l=1}^{\infty} (\frac{1}{10})^{l^2} (g_l(x) - g_l(x+e_1)) = \\ &= \sum_{l=N}^{\infty} (\frac{1}{10})^{l^2} (g_l(x) - g_l(x+e_1)) \geq \\ &\geq (\frac{1}{10})^{N^2} - \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2} |g_l(x) - g_l(x+e_1)| \geq \\ &\geq (\frac{1}{10})^{N^2} - \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2} 3^{l+1} = \\ &= (\frac{1}{10})^{N^2} - (\frac{1}{10})^{N^2} \sum_{l=N+1}^{\infty} (\frac{1}{10})^{l^2-N^2} 3^{l+1} > (\frac{1}{10})^{N^2} - \frac{1}{10}(\frac{1}{10})^{N^2} > 0 \end{aligned} \tag{2.18}$$

by (2.16). (2.1)(iii) is proven. We have constructed $g : \mathbf{Z}^2 \to \mathbf{R}$ satisfying (2.1).

3. Constructing a Continuous Function

Let us extend g to a function G on all of $\mathbf{R}^2$ that is almost periodic and has no local minimum. We will define $G(x)$ to be a weighted sum of $g(\xi)$ for several $\xi \in \mathbf{Z}^2$ close to x. Let the function $\psi \in C^\infty(\mathbf{R}^2, [0,1])$ be supported on the square $[-1,1] \times [-1,1]$; that is,

$$\psi(x) \neq 0 \Rightarrow |x_1| < 1 \text{ and } |x_2| < 1. \tag{3.0}$$

We will define ψ more precisely later. Define $G : \mathbf{R}^2 \to \mathbf{R}$ by

$$G(x) = \sum_{\xi \in \mathbf{Z}^2} \psi(x - \xi) g(\xi). \tag{3.1}$$

Since G can be regarded as the convolution of a C_0^∞ function with a measure, G is well-defined and infinitely differentiable. To prove that G is almost periodic, let $\epsilon > 0$. Let $\mathcal{A}$ be a relatively dense set of $\epsilon/4$-almost periods of g, in the sense of Section 2. Let $a \in \mathcal{A}$ and $x \in \mathbf{R}^2$. Then

$$\begin{aligned} G(x) - G(x+a) &= \sum_{\xi \in \mathbf{Z}^2} \psi(x-\xi) g(\xi) - \sum_{\xi \in \mathbf{Z}^2} \psi((x+a)-\xi) g(\xi) = \\ &= \sum_{\xi \in \mathbf{Z}^2} \psi(x-\xi) g(\xi) - \sum_{\xi \in \mathbf{Z}^2} \psi((x+a)-(\xi+a)) g(\xi+a) = \\ &= \sum_{\xi \in \mathbf{Z}^2} \psi(x-\xi) g(\xi) - \sum_{\xi \in \mathbf{Z}^2} \psi(x-\xi) g(\xi+a) = \\ &= \sum_{\xi \in \mathbf{Z}^2} \psi(x-\xi)(g(\xi) - g(\xi+a)). \end{aligned} \tag{3.2}$$

By (3.0) the last summation has no more than four nonzero terms. Therefore, since $0 \leq \psi \leq 1$,

$$|G(x) - G(x+a)| \leq \sum_{\xi \in \mathbf{Z}^2} |g(\xi) - g(\xi+a)| < 4(\frac{\epsilon}{4}) = \epsilon. \tag{3.3}$$

G is almost periodic.

To show that G has no local minimum, let us define ψ more precisely. Let φ satisfy

$$\begin{aligned} &(i)\ \varphi \in C^\infty(\mathbf{R}, [0,1]) \\ &(ii)\ |t| \geq 1 \Rightarrow \varphi(t) = 0 \\ &(iii)\ \varphi(0) = 1 \\ &(iv)\ \varphi(-t) = \varphi(t) \text{ for all } t \in \mathbf{R} \\ &(v)\ 0 < t < 1 \Rightarrow \varphi(t) + \varphi(1-t) = 1 \\ &(vi)\ 0 < t < 1 \Rightarrow \varphi'(t) < 0 \end{aligned} \tag{3.4}$$

Now define

$$\psi(x) = \varphi(x_1)\varphi(x_2). \tag{3.5}$$

This is consistent with (3.0). Let G be defined as in (3.1). We will prove that for any $x \in \mathbf{R}^2$, x is not a local minimum of G. There are three cases: (i) $x \in \mathbf{Z}^2$, (ii) x is not in $\mathbf{Z}^2$, but belongs to a horizontal or vertical segment connecting two points in $\mathbf{Z}^2$, and (iii) x has the form $x = (\xi_1 + s, \xi_2 + t)$ for some $\xi = (\xi_1, \xi_2) \in \mathbf{Z}^2$ and $s, t \in (0, 1)$.

If $x \in \mathbf{Z}^2$, then $G(x) = g(x)$ by (3.4)(ii)-(iii). By (2.1)(iii), $g(x + \hat{e}) < g(x)$ for some $\hat{e} \in \{e_1, -e_1, e_2, -e_2\}$. We will show that on the line segment connecting x and $x + \hat{e}$, G behaves like φ, that is, for $t \in [0, 1]$,

$$G(x + t\hat{e}) = g(x) - (1 - \varphi(t))(g(x) - g(x + \hat{e})). \tag{3.6}$$

Therefore x is not a local minimum of G, since by (3.4)(vi), $G(x + t\hat{e}) < G(x)$ for all $t \in (0, 1]$. We will prove (3.6) for when $\hat{e} = e_1$ or $-e_2$; the other two cases are similar. First suppose $\hat{e} = e_1$. For all $t \in [0, 1]$, the open square $((x_1 + t) - 1, (x_1 + t) + 1) \times (x_2 - 1, x_2 + 1)$ contains no points of $\mathbf{Z}^2$ other than (possibly) x and $x + e_1$. Therefore $G(x + te_1)$ has the form

$$\begin{aligned} G(x + te_1) &= \psi((x + te_1) - x)g(x) + \psi((x + te_1) - (x + e_1))g(x + e_1) = \\ &= \psi(te_1)g(x) + \psi((t - 1)e_1))g(x + e_1) = \\ &= \varphi(t)g(x) + \varphi(t - 1)g(x + e_1) = \varphi(t)g(x) + \varphi(1 - t)g(x + e_1) = \end{aligned} \tag{3.7}$$

(by (3.4)(iv))

$$= \varphi(t)g(x) + (1 - \varphi(t))g(x + e_1) = g(x) - (1 - \varphi(t))(g(x) - g(x + e_1))$$

by (3.4)(v). This proves (3.6) for the case $\hat{e} = e_1$. If, instead, $\hat{e} = -e_2$, then

$$\begin{aligned} G(x - te_2) &= \psi((x - te_2) - x)g(x) + \psi((x - te_2) - (x - e_2))g(x - e_2) = \\ &= \psi(-te_2)g(x) + \psi((1 - t)e_2))g(x - e_2) = \\ &= \varphi(-t)g(x) + \varphi(1 - t)g(x - e_2) = \varphi(t)g(x) + \varphi(1 - t)g(x - e_2) = \\ &= \varphi(t)g(x) + (1 - \varphi(t))g(x - e_2) = g(x) - (1 - \varphi(t))(g(x) - g(x - e_2)). \end{aligned} \tag{3.8}$$

(3.6) is proven.

The second case occurs when x has the form $\xi + t\hat{e}$ for some $\xi \in \mathbf{Z}^2$, $t \in (0,1)$ and $\hat{e} = e_1$ or e_2. By (2.1)(ii), $g(\xi) \neq g(\xi + \hat{e})$. If $g(\xi) > g(\xi + \hat{e})$, then (3.6) implies that $G(\xi + t'\hat{e}) < G(\xi + t\hat{e}) = G(x)$ for $t' \in (t, 1]$. If $g(\xi) < g(\xi + \hat{e})$, then (3.6) implies that $G(x + t'\hat{e}) < G(\xi + t\hat{e})$ for $t' \in [0, t)$. In each case, x is not a local minimum of G.

The third and final case occurs when x has the form $x = (\xi_1 + s, \xi_2 + t)$ for some $\xi = (\xi_1, \xi_2) \in \mathbf{Z}^2$ and $s, t \in (0,1)$. For convenience assume that $\xi = (0,0)$. Let $x \in (0,1) \times (0,1)$. If x is a local minimum of G, x must be a critical point of G. We will assume x is a critical point of G and show that x is not a local minimum. For any y in the open unit square $(0,1) \times (0,1)$, $\psi(y - \xi)$ equals zero for all $\xi \in \mathbf{Z}^2$ except for the four points $\xi = (0,0), e_1, e_2$, and $e_1 + e_2$. Therefore $G(y)$ has the form

$$
\begin{aligned}
(3.9)\quad G(y) &= \psi(y)g(0,0) + \psi(y - e_1)g(e_1) + \psi(y - e_2)g(e_2) + \\
&\quad + \psi(y - e_1 - e_2)g(e_1 + e_2) = \\
&= \varphi(y_1)\varphi(y_2)g(0,0) + \varphi(y_1 - 1)\varphi(y_2)g(e_1) + \\
&\quad + \varphi(y_1)\varphi(y_2 - 1)g(e_2) + \varphi(y_1 - 1)\varphi(y_2 - 1)g(e_1 + e_2) = \\
&= \varphi(y_1)\varphi(y_2)g(0,0) + \varphi(1 - y_1)\varphi(y_2)g(e_1) + \\
&\quad + \varphi(y_1)\varphi(1 - y_2)g(e_2) + \varphi(1 - y_1)\varphi(1 - y_2)g(e_1 + e_2) =
\end{aligned}
$$

(by (3.4)(iv))

$$
\begin{aligned}
&= \varphi(y_1)\varphi(y_2)g(0,0) + (1 - \varphi(y_1))\varphi(y_2)g(e_1) + \\
&\quad + \varphi(y_1)(1 - \varphi(y_2))g(e_2) + (1 - \varphi(y_1))(1 - \varphi(y_2))g(e_1 + e_2) =
\end{aligned}
$$

(by (3.4)(v))

$$
\begin{aligned}
&= g(1,1) + [g(0,1) - g(1,1)]\varphi(y_1) + [g(1,0) - g(1,1)]\varphi(y_2) + \\
&\quad + [g(0,0) - g(1,0) - g(0,1) + g(1,1)]\varphi(y_1)\varphi(y_2).
\end{aligned}
$$

Let $x \in (0,1) \times (0,1)$ be a critical point of G. By (3.9), for $t \in [0,1]$, $G(x_1, t)$ has the form

$$
(3.10)\qquad G(x_1, t) = A + B\varphi(t)
$$

for some real numbers A and B that depend on x_1. Since x is a critical point of G, $\frac{\partial}{\partial x_2}G(x) = 0 = B\varphi'(x_2)$. By (3.4)(vi), $\varphi'(x_2) \neq 0$, so $B = 0$, and G is constant along the segment connecting $(x_1, 0)$ and $(x_1, 1)$, that is,

$$
(3.11)\qquad G(x_1, t) = G(x) \equiv G(x_1, x_2)
$$

for all $t \in [0,1]$. By similar reasoning, there exist numbers A and B such that

$$G(s, x_2) = A + B\varphi(s) \tag{3.12}$$

for all $s \in (0,1)$. Since $\frac{\partial}{\partial x_1}G(x) = 0$, we obtain $B = 0$ as above, so G is constant along the segment connecting $(0, x_2)$ and $(1, x_2)$, that is,

$$G(s, x_2) = G(x) \equiv G(x_1, x_2) \tag{3.13}$$

for all $s \in [0,1]$.

Let $\epsilon > 0$ with

$$\epsilon < \min\{x_1, x_2, 1 - x_1, 1 - x_2\}. \tag{3.14}$$

Then $G(x_1 + \epsilon, x_2) = G(x_1, x_2)$. As above, there exist constants A and B such that

$$G(x_1 + \epsilon, t) = A + B\varphi(t) \tag{3.15}$$

for all $t \in [0,1]$. We claim $B \neq 0$: if $B = 0$, then G is constant along the vertical segment $\{x_1 + \epsilon\} \times [0,1]$. Thus (3.11) and (3.13) give

$$G(x_1, 1) = G(x_1, x_2) = G(x_1 + \epsilon, x_2) = G(x_1 + \epsilon, 1). \tag{3.16}$$

This is impossible because by (3.6), for $s \in [0,1]$, $G(s,1)$ has the form

$$G(s, 1) = g(0,1) - (1 - \varphi(s))(g(0,1) - g(1,1)). \tag{3.17}$$

By (2.1)(ii), $g(0,1) \neq g(1,1)$, and by (3.4)(vi), $\varphi(x_1) \neq \varphi(x_1 + \epsilon)$. Therefore the above gives $G(x_1, 1) \neq G(x_1 + \epsilon, 1)$, contradicting (3.16). Therefore $B \neq 0$ in (3.15).

If $B > 0$ in (3.15), then by (3.4)(vi),

$$G(x_1 + \epsilon, x_2 + \epsilon) = A + B\varphi(x_2 + \epsilon) < A + B\varphi(x_2) = G(x_1 + \epsilon, x_2) = G(x_1, x_2). \tag{3.18}$$

If $B < 0$ in (3.15), then

$$G(x_1 + \epsilon, x_2 - \epsilon) = A + B\varphi(x_2 - \epsilon) < A + B\varphi(x_2) = G(x_1 + \epsilon, x_2) = G(x_1, x_2). \tag{3.19}$$

In either case, we have found a point at distance $\sqrt{2}\epsilon$ from x where the value of G is less than $G(x)$. Since ϵ can be arbitrarily small, x is not a local minimum of G.

Open Questions

Some unanswered questions remain. The function G constructed here has an absolute maximum at $(0,0)$. Does there exist an almost periodic function on $\mathbf{R}^n$ with no local minimum *or* maximum? Does there exist a real analytic, almost periodic function on $\mathbf{R}^n$ with no local minimum? It is not even obvious that a continuously differentiable, almost periodic function on $\mathbf{R}^n$ must have a critical point. Finally, there are periodicity conditions that are weaker than periodic and stronger than almost periodic, such as quasiperiodic. These notions are easily formulated for functions on $\mathbf{R}^n$. Does a quasiperiodic function on $\mathbf{R}^n$ have to have a local minimum?

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