An Elliptic Partial Differential Equation with a Symmetrical Almost Periodic Term

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1. Introduction

In [STT], a Hamiltonian system of the form

$$-u'' + u = h(t)\nabla F(u) \tag{1.0}$$

was studied, where h is an almost periodic (defined in a moment) function, and $F : \mathbf{R}^N \to \mathbf{R}$ a "superquadratic" potential. That is, $F(q)$ behaves like q to a power greater than 2, with $F(q)/|q|^2 \to 0$ as $|q| \to 0$ and $F(q)/|q|^2 \to \infty$ as $|q| \to \infty$. For example, $F(q) = |q|^{p-1}q$ with $p > 1$ would qualify. The authors found that (1.0) must have a nonzero solution homoclinic to zero. Since this result, many papers (see [CMN], [R1], and [ACM], for example) have been written concerning Hamiltonian systems with almost periodic terms.

"Almost periodic" is defined for functions on $\mathbf{R}^N$ $(N > 1)$ and even on more general topological groups. Thus one can write a PDE version of (1.0),

$$-\Delta u + u = h(x)f(u), \tag{1.1}$$

wherein h is almost periodic and the primitive F of f satisfies appropriate superquadraticity and critical growth conditions. Then a natural question is, does (1.1) have a nontrivial "homoclinic-type" solution, that is, a solution u with $|\nabla u(x)| + |u(x)| \to 0$ as $m \to \infty$? In [S1] we took a step toward answering in the affirmative. We considered an equation of the form $-\epsilon^2\Delta u + V(x)u = f(u)$, with V almost periodic and f as above. It was shown that if V satisfied an additional condition, then for small enough ϵ, the equation has a homoclinic-type solution. That condition was automatically satisfied when $N = 2$. In this paper, we obtain a similar result for equation (1.1), that is, an equation without the ϵ. We will impose an extra symmetry condition on h, however.

Let us define an almost periodic function on $\mathbf{R}^N$ ($\mathbf{R}$ is a special case, and defining an a.p. function on other topological groups is an obvious generalization). First, a set $\mathcal{A} \subset \mathbf{R}^N$ is *relatively dense* if there exists $L > 0$ such that for every $x \in \mathbf{R}^N$, there exists $y \in \mathcal{A}$ with $|x - y| < L$. Next, for $\epsilon > 0$, $\vec{v} \in \mathbf{R}^N$, and $h : \mathbf{R}^N \to \mathbf{R}$, we say $\vec{v}$ is an *ϵ-almost period* of h if for all $x \in \mathbf{R}^N$, $|h(x + \vec{v}) - h(x)| < \epsilon$. Finally, h is defined to be *almost periodic* if h is continuous and for every $\epsilon > 0$, there exists a relatively dense set $\mathcal{A} \equiv \mathcal{A}(\epsilon) \subset \mathbf{R}^N$ such that for all $a \in \mathcal{A}$, a is an ϵ-almost period of h. For properties of almost periodic functions (many properties of a.p. functions on $\mathbf{R}$ extend to functions on $\mathbf{R}^N$), see [Be], [Bo], [C], [Z].

We will prove the following:

THEOREM 1.2 *Let h and f satisfy*

(h_1) $h \in C(\mathbf{R}^N, \mathbf{R})$,

(h_2) $\inf_{\mathbf{R}^N} h > 0$

(h_3) *h is almost periodic.*

(h_4) $h(x_1, \ldots, -x_i, \ldots, x_N) = h(x_1, \ldots, x_i, \ldots, x_N)$ *for all* $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$ *and all* $i = 1, \ldots, N - 1$.

(f_1) $f \in C^1(\mathbf{R}^+, \mathbf{R})$

(f_2) $f'(0) = 0$.

(f_3) *There exist $A, s > 0$ such that $|f'(q)| \le A(1 + |q|^{s-1})$ for all $q \ge 0$. If $n \ge 3$, then* $s < 4/(n - 2)$.

(f_4) *For some $\mu > 2$, $0 < \mu F(q) \le f(q)q$ for all $q > 0$, where $F(q) \equiv \int_0^q f(t)\, dt$.*

(f_5) *The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.*

Then equation (1.1) has a positive homoclinic-type solution. More precisely, if c_0 is the mountain-pass value of the functional I associated with (1.1), then for each $\epsilon > 0$, there exists a positive solution u of (1.1) with $c_0 \le I(u) < c_0 + \epsilon$.

In a moment we will give a precise definition of c_0. An example of f satisfying $(f_1) - (f_5)$ is $f(q) = q^s$, where s is as in (f_3). Condition (f_5) is an important convexity condition found in many papers, such as [R2], [WZ], and [FdP1-3]. Condition (h_4) states that h is even with respect to the hyperplane $\{x_i = 0\}$ for all $i = 0, \ldots, N - 1$ (but not necessarily for $i = N$). An example of h satisfying $(h_1) - (h_4)$ is $h(x_1, x_2) = 6 + \cos(x_1) + \cos(\sqrt{2}x_1) + \sin(x_2) + \sin(\sqrt{2}x_2) + \sin(\sqrt{6}x_2)$. This h is even with respect to the x_2-axis, but not the x_1-axis.

Although the symmetry assumption (h_4) is a strong one, proving Theorem 1.2 is not easy. The proof of [STT]'s $N = 1$ result cannot be directly generalized (see [S1] for explanation). Also, an almost periodic function of several variables need not have a local minimum (see [S2]), ruling out some promising variational approaches.

Note that other than symmetry, there are few extra restrictions on h; for example, no assumptions on the variation of h are necessary (as in [S3]), nor on the magnitude of $|\nabla h|$ (if defined, or on a modulus of continuity for h if h is not C^1).

Variational Framework and Plan of Proof

Define $E = W^{1,2}(\mathbf{R}^N)$. Extend f and F to $\mathbf{R}$ by defining $f(-q) = f(q)$. Define a C^2 functional I on E by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} h(x)F(u)\,dx. \tag{1.3}$$

Then, by elliptic regularity theory, the set of critical points of I equals the set of homoclinic-type solutions of (1.1). We will find a nonzero critical point of I, then show it is a positive function. Define

$$\Gamma_0 = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\} \tag{1.4}$$

and

$$c_0 = \inf_{\gamma\in\Gamma} \max_{\theta\in[0,1]} I(\gamma(\theta)). \tag{1.5}$$

This is the c_0 from the statement of Theorem 1.2. I satisfies most of the hypotheses of the Mountain Pass Theorem ([AR]), so $c_0 > 0$ (see [CR1] for a similar example). For $i = 1, \ldots, N-1$, define the reflection operator $R_i : \mathbf{R}^N \to \mathbf{R}^N$ by

$$R_i(x_1, \ldots, x_i, \ldots, x_N) = (x_1, \ldots, -x_i, \ldots, x_N), \tag{1.6}$$

Also define $R_i : E \to E$ by

$$(R_i u)(x) = u(R_i x). \tag{1.7}$$

For $\{i_1, i_2, \ldots, i_k\} \subset \{1, \ldots, N-1\}$, define $E[i_1, i_2, \ldots, i_k] \subset E$ by

$$E[i_1, i_2, \ldots, i_k] = \{u \in E \mid R_i u = u \text{ for } i = i_1, i_2, \ldots, i_k\}, \tag{1.8}$$

$\Gamma[i_1, i_2, \ldots, i_k] \subset \Gamma$ by

$$(1.9) \qquad \Gamma[i_1, i_2, \ldots, i_k] = \{\gamma \in \Gamma \mid \gamma(\theta) \in E[i_1, i_2, \ldots, i_k] \text{ for all } \theta \in [0,1]\},$$

and

$$(1.10) \qquad c[i_1, i_2, \ldots, i_k] = \inf_{\gamma \in \Gamma[i_1, i_2, \ldots, i_k]} \max_{\theta \in [0,1]} I(\gamma(\theta)).$$

Finally, for $k = 1, 2, \ldots, N-1$, define

$$(1.11) \qquad c_k = \min \{c[i_1, i_2, \ldots, i_k] \mid \{i_1, i_2, \ldots, i_k\} \subset \{1, \ldots, N-1\}\}$$

and

$$(1.12) \qquad E_k = \bigcup \{E[i_1, i_2, \ldots, i_k] \mid i_1, i_2, \ldots, i_k \subset \{1, \ldots, N-1\}\}$$

Then $c_0 \leq c_1 \leq \ldots \leq c_{N-1}$. We will prove Theorem 1.2 by proving the following two propositions:

Proposition 1.13 *If $c_{N-1} = c_0$, then for any $\epsilon > 0$, I has a critical point u with $c_0 \leq I(u) < c_0 + \epsilon$.*

Proposition 1.14 *If there exists $d > 0$ such that the interval $[c_0, c_0 + d)$ contains no critical values of I, then for $i = 0, 1, \ldots, N-2$, $c_i = c_0$ implies $c_{i+1} = c_0$.*

The proof of Theorem 1.2 then runs as follows: Let $\epsilon > 0$. If there does not exist a critical point u of I with $c_0 \leq I(u) < c_0 + \epsilon$, then by Proposition 1.14, $c_0 = c_1 = \ldots = c_{N-1}$. Thus by Proposition 1.13, I has a critical level in the interval $[c_0, c_0 + \epsilon)$. This is a contradiction. Therefore (1.1) has a homoclinic-type solution u_ϵ with $c_0 \leq I(u_\epsilon) < c_0 + \epsilon$. We then show that if ϵ is small enough, u_ϵ does not change sign.

Organization of Paper

Section 2 contains some technical results. Section 3 contains the proofs of Propositions 1.13 and 1.14.

2. Technical Results

There is another definition of almost periodic that is equivalent ([Be]) to that given in the Introduction. For $h : \mathbf{R}^N \to \mathbf{R}$ and $x \in \mathbf{R}^N$, define $\tau_x h$ by $\tau_x h(y) = h(y-x)$. That is, $\tau_x h$ is h translated by x. Then, a continuous function $h : \mathbf{R}^N \to \mathbf{R}$ is almost periodic if the set of its translates is precompact under the uniform norm. That is, h is almost periodic if, for any sequence $(x_m) \subset \mathbf{R}^N$, there is a subsequence $(y_m) \subset (x_m)$ and a continuous function $\bar{h} : \mathbf{R}^N \to \mathbf{R}$ with $\tau_{y_m} h \to \bar{h}$ uniformly on $\mathbf{R}^N$ as $m \to \infty$. For almost periodic h, define the *hull* of h, $Hull(h)$, to be the set of all limit points of $\{\tau_x h \mid x \in \mathbf{R}^N\}$ under the topology of uniform convergence.

I fails the Palais-Smale condition. That is, a *Palais-Smale sequence*, or sequence (u_m) with $I'(u_m) \to 0$ and $(I(u_m))$ convergent, need not be precompact. The following concentration compactness result, however, describes Palais-Smale sequences of I. For $\bar{h} : \mathbf{R}^N \to \mathbf{R}$, define $I[\bar{h}]$ by $I[\bar{h}](u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} \bar{h}(x) F(u(x))\, dx$. In particular, $I = I[h]$.

Proposition 2.0 *Let h and f satisfy $(h_1) - (h_3)$ and $(f_1) - (f_4)$. Let $(u_m) \subset E$ with $I'(u_m) \to 0$ and $I(u_m) \to b > 0$ as $m \to \infty$. Then there exists a subsequence of (u_m) (also denoted (u_m)), $k \in \mathbf{N}$, sequences $(x_m^i)_{m\geq 1}^{i=1,\ldots,k}$, functions $(\bar{h}_i)^{i=1,\ldots,k} \subset Hull(h)$, and $(v_i)^{i=1,\ldots,k} \subset E \setminus \{0\}$ satisfying*

(i) $|x_m^i - x_m^j| \to \infty$ *for* $i \neq j$ *as* $m \to \infty$

(ii) $\tau_{-x_m^i} h \to \bar{h}_i$ *uniformly as* $m \to \infty$

(iii) $I[\bar{h}_i]'(v_i) = 0$ *for all* i

(iv) $\|u_m - \sum_{i=1}^{k} \tau_{x_m^i} v_i\| \to 0$ *as* $m \to \infty$

(v) $\sum_{i=1}^{k} I[\bar{h}_i](v_i) = b.$

This is essentially a special case of a result in [S1]. The proof follows that of a similar result in [CR1-2], using principles found in [L].

The above result does not require (f_5). If we assume (f_5), then for any $u \neq 0$, the mapping $t \mapsto I(tu)$ is increasing for small positive t, achieves a maximum at some $t' > 0$, then decreases to infinity on (t', ∞) (see [CR1]). Therefore, if $u \neq 0$ with $I'(u) = 0$, or even just $I'(u)u = 0$, then $I(u) \geq c_0$, where c_0 is the minimax value defined in (1.5). If we define $c(\bar{h})$ to be the minimax value associated with the functional $I[\bar{h}]$, then $c(\bar{h}) = c_0$ for all $\bar{h} \in Hull(h)$ (see [CMN]). Therefore, Proposition 2.0, in particular Proposition 2.0(v), gives the following strong result:

PROPOSITION 2.1 *Let h and f satisfy $(h_1) - (h_3)$ and $(f_1) - (f_5)$. Let $(u_m) \subset E$ with $I'(u_m) \to 0$ and $I(u_m) \to b \in (0, 2c_0)$ as $m \to \infty$. Then there exists a subsequence of (u_m) (also denoted (u_m)), a sequence $(x_m) \subset \mathbf{R}^N$, a function $\bar{h} \in Hull(h)$, and $v \in E$ satisfying*

(i) $\tau_{-x_m} h \to \bar{h}$ *uniformly as* $m \to \infty$

(ii) $I[\bar{h}]'(v) = 0$

(iii) $\|u_m - \tau_{x_m} v\| \to 0$ *as* $m \to \infty$

(iv) $I[\bar{h}](v) = b$

Define the "solution manifold" or Nehari manifold $\mathcal{S} \subset E$ by $\mathcal{S} = \{u \in E \setminus \{0\} \mid I'(u)u = 0\}$. Because of (f_5), $\mathcal{S}$ is homeomorphic to the unit sphere in E via a radial map, and

$$c_0 = \inf_{\mathcal{S}} I. \tag{2.2}$$

(see [R2]). We can strengthen this result to include sequences which are "close to $\mathcal{S}$." That is, the following is true:

PROPOSITION 2.3

$$c_0 = \inf\{\lim \inf_{m\to\infty} I(u_m) \,|(u_m) \subset E,\ 0 < \lim \inf_{m\geq 1} I(u_m) \leq \lim \sup_{m\geq 1} I(u_m) < \infty, \\ I'(u_m)u_m \to 0\}. \tag{2.4}$$

This is essentially the same result as Proposition 3 in [CT], extended to a PDE setting. In that paper, the authors worked with a Palais-Smale sequence (u_m), with $I'(u_m) \to 0$. However, their proof did not use the full strength of that assumption, which could have been replaced by $I'(u_m)u_m \to 0$. The distinction is important to us, for we will be working with sequences (u_m) which satisfy $I'(u_m)u_m \to 0$ but not, apparently, $I'(u_m) \to 0$.

Proof of Proposition 2.3: the proof is similar to that in [CT]. The direction $c_0 \geq \dots$ is obvious by (2.2). To prove the other inequality, let $(u_m) \subset E$ be as in (2.4). By arguments of [CMN], for example, (u_m) is bounded in E. By arguments of [CR2], Lemma 2.25 (these arguments assume $I'(u_m) \to 0$ but require only $I'(u_m)u_m \to 0$), there exist $\rho > 0$ and a sequence $(x_m) \subset \mathbf{R}^N$ such that $\|u_m\|_{L^2(B_1(x_m))} > \rho$ for large m. Since (u_m) is bounded, there exists $v \in E$ such that, along a subsequence, $\tau_{-x_m} u_m \rightharpoonup v$ weakly in E as $m \to \infty$ and also in L^p_{loc} for $1 \leq p < 2n/(n-2)$. $\|\tau_{-x_m} u_m\|_{L^2(B_1(0))} > \rho$ for large m, so $v \neq 0$.

Let $\Omega \subset \mathbf{R}^N$ be bounded and have positive Lebesgue measure, with $|v| > 2\delta > 0$ a.e. on Ω. Since $\tau_{-x_m} u_m \to v$ in $L^2(\Omega)$, there exists $a > 0$ with

$$\lambda(\{x \mid |u_m(x)| > \delta\}) \geq \lambda(\{x \mid |u_m(x)| > \delta\} \cap \Omega) > a \tag{2.5}$$

for large enough m, where λ denotes Lebesgue measure.

Define ρ_m to be the unique positive number satisfying $\rho_m u_m \in \mathcal{S}$. Let $h_- = \inf_{\mathbf{R}^N} h > 0$. Then for large m, $I'(\rho_m u_m)(\rho_m u_m) = 0 = I'(\rho_m u_m)(u_m)$, so

$$\begin{aligned} \|u_m\|^2 &= \frac{1}{\rho_m}\int_{\mathbf{R}^N} hf(\rho_m u_m)u_m \geq \frac{h_-}{\rho_m}\int_{\{|u_m|>\delta\}} f(\rho_m u_m)u_m = \\ &= h_- \int_{\{|u_m|>\delta\}} \frac{f(\rho_m u_m)}{\rho_m u_m} u_m^2 \geq h_- \delta^2 \int_{\{|u_m|>\delta\}} \frac{f(\delta\rho_m)}{\delta\rho_m} = \\ &\text{(by } (f_5)) \\ &= h_-\delta^2 \frac{f(\delta\rho_m)}{\delta\rho_m}\lambda(\{|u_m| > \delta\}) > (h_-\delta^2 a)\left(\frac{f(\delta\rho_m)}{\delta\rho_m}\right). \end{aligned} \tag{2.6}$$

Since $f(q)/q \to \infty$ as $|q| \to \infty$, and $(\|u_m\|)$ is bounded, it follows that (ρ_m) is bounded. Define $g_m, h_m : \mathbf{R}^+ \to \mathbf{R}$ by

$$g_m(\rho) = I(\rho u_m) = \frac{1}{2}\rho^2\|u_m\|^2 - \int_{\mathbf{R}^N} hF(\rho u_m) \tag{2.7}$$

and

$$h_m(\rho) = \frac{1}{2}\rho^2 \int_{\mathbf{R}^N} hf(u_m)u_m - \int_{\mathbf{R}^N} hF(\rho u_m). \tag{2.8}$$

(f_5) gives $h_m' > 0$ on $(0,1)$ and $h_m' < 0$ on $(1,\infty)$. Now

$$g_m(\rho_m) - h_m(\rho_m) = \frac{1}{2}\rho_m^2(\|u_m\|^2 - \int_{\mathbf{R}^N} hf(u_m)u_m) = \frac{1}{2}\rho_m^2 I'(u_m)u_m. \tag{2.9}$$

Since (ρ_m) is bounded and $I'(u_m)u_m \to 0$, $g_m(\rho_m) - h_m(\rho_m) \to 0$. The same argument shows that $g_m(1) - h_m(1) \to 0$. Therefore

$$\begin{aligned}(2.10)\qquad c_0 &\le \lim_{m\to\infty}\inf I(\rho_m u_m) \equiv \lim_{m\to\infty}\inf g_m(\rho_m) = \lim_{m\to\infty}\inf h_m(\rho_m) \le \\ &\le \lim_{m\to\infty}\inf h_m(1) = \lim_{m\to\infty}\inf g_m(1) \equiv \lim_{m\to\infty}\inf I(u_m).\end{aligned}$$

The direction $c_0 \le \ldots$ in (2.4) is proven.

The proof of (2.2) can be adapted easily to show

$$(2.11)\qquad c_k = \inf_{\mathcal{S}\cap E_k} I,$$

where E_k is defined in (1.12). Therefore, the proof of Proposition 2.3 shows that for $k = 0, 1, \ldots, N-1$,

$$\begin{aligned}(2.12)\qquad c_k = \inf\{\lim_{m\to\infty}\inf I(u_m)\ |(u_m) \subset E_k,\ & \\ & 0 < \lim_{m\to\infty}\inf I(u_m) \le \lim_{m\to\infty}\sup I(u_m) < \infty, \\ & I'(u_m)u_m \to 0\}.\end{aligned}$$

3. Proofs of Propositions 1.13 and 1.14

Proof of Proposition 1.13

Suppose $c_{N-1} = c_0$. Let $\epsilon > 0$ and let $\gamma \in \Gamma[1, 2, \ldots, N-1]$ with

$$(3.0)\qquad \max_{\theta\in[0,1]} I(\gamma(\theta)) < c_0 + \epsilon.$$

Let V denote the gradient of I. That is, $(V(u), w) = I'(u)w$ for all $u, w \in E$. Let η be the solution of the initial value problem $\frac{d\eta}{dt} = -V(\eta)$; $\eta(0, u) = u$. By arguments in [CMN] or [S1], for all $u \in E$, either $\eta(t, u)$ is well-defined and $I(\eta(t, u)) \ge 0$ for all $t > 0$, or $I(\eta(t, u)) < 0$ for some $t > 0$. Because of the mountain-pass structure of I, it is apparent that for some $\theta \in [0, 1]$, $\lim_{t\to\infty} I(\eta(t, \gamma(\theta))) \in [c_0, c_0 + \epsilon)$. By Lemma 3.1 of [CMN], $I'(\eta(t, \gamma(\theta))) \to 0$ as $t \to \infty$. From here on, denote $\eta \equiv \eta(t) \equiv \eta(t, \gamma(\theta))$.

Let $u_m = \eta(m)$. Since $I'(\eta(t)) \to 0$ as $t \to \infty$, $\|u_{m+1} - u_m\| \to 0$ as $m \to \infty$. Using Proposition 2.0, together with an indirect argument, it is easy to show that there exists $\rho > 0$ and a sequence $(y_m) \subset \mathbf{R}^N$ with $\|u_m\|_{W^{1,2}(B_1(y_m))} > \rho$ for large m. Let $\mathbf{e}_i = (0, 0, \ldots, 1, \ldots, 0)$ denoted the ith standard basis vector for $\mathbf{R}^N$. We claim that for $i \leq N-1$, $(y_m \cdot \mathbf{e}_i)$ is bounded in m. Proof: suppose not. Then $(y_m \cdot \mathbf{e}_i)$ has an unbounded subsequence for, say, $i = 1$. Let $(u_{m_i}) \subset (u_m)$ be a subsequence along which $|y_{m_i} \cdot \mathbf{e}_1| \to \infty$. Without loss of generality assume $\epsilon < c_0$. Apply Proposition 2.1 to (u_{m_i}) to obtain a subsequence (also denoted (u_{m_i})), a sequence $(x_i) \subset \mathbf{R}^N$, and $v \in E$ with $\|u_{m_i} - \tau_{x_i} v\| \to 0$ as $i \to \infty$. Then

$$\|\tau_{x_i} v\|_{W^{1,2}(B_1(y_{m_i}))} > \rho/2 \tag{3.1}$$

for large i. The flow η preserves symmetry with respect to the x_1-, $x_2-, \ldots, x_{N-1}-$ hyperplanes. Therefore $u_m \in E[1, 2, \ldots, N-1]$ for all m, using the notation of (1.8). So for large i, $\|u_{m_i}\|_{W^{1,2}(B_1(R_1(y_{m_i})))} > \rho$, and

$$\|\tau_{x_i} v\|_{W^{1,2}(B_1(R_1(y_{m_i})))} > \rho/2. \tag{3.2}$$

By (3.1)-(3.2),

$$\|v\|_{W^{1,2}(B_1(y_{m_i} - x_i))} > \rho/2 \text{ and } \|v\|_{W^{1,2}(B_1(R_1(y_{m_i}) - x_i))} > \rho/2. \tag{3.3}$$

This is impossible because $|y_{m_i} \cdot \mathbf{e}_1| \to \infty$ as $i \to \infty$, implying $|y_{m_i} - R_1(y_{m_i})| \to \infty$ and $|y_{m_i} - x_i| + |R_1(y_{m_i}) - x_i| \to \infty$. The claim is proven, and for all $i \leq N-1$, $(y_m \cdot \mathbf{e}_i)$ is bounded in m.

Next, we claim that $y_m - y_{m+1}$ is bounded in m. To prove, assume otherwise. Then along a subsequence $(y_{m_i}) \subset (y_m)$, $|y_{m_i+1} - y_{m_i}| \to \infty$. Apply Proposition 2.1 to (y_{m_i}). By a similar argument to that above, we obtain a contradiction.

$(y_m \cdot \mathbf{e}_N)$ is either bounded or unbounded in m. If bounded, then (y_m) is bounded. Taking a subsequence, we may assume $y_m \to \bar{y}$ as $m \to \infty$. Applying Proposition 2.1 to (u_m), we obtain a subsequence (also called (u_m)), $(x_m) \subset \mathbf{R}^N$, $\bar{h} \in Hull(h)$ with $\tau_{-x_m} h \to \bar{h}$ uniformly, and v with $\|u_m - \tau_{x_m} v\| \to 0$, $I[\bar{h}]'(v) = 0$, and $I[\bar{h}](v) \in [c_0, c_0 + \epsilon)$. It suffices to show that $\bar{h}$ is merely a translate of h. By arguments similar to those above, it is easy to show that $(x_m - y_m)$ is bounded. Therefore, we assume that (x_m) is convergent. Then obviously $\bar{h}$ is a translate of h.

The other alternative is that $(y_m \cdot \mathbf{e}_N)$ is unbounded in m. By [S1], for any $\delta > 0$, there exists a relatively dense subset $\mathcal{A} \subset \mathbf{R}$ such that for all $a \in \mathcal{A}$, $a\mathbf{e}_N$ is a δ-almost period of h. Since $(y_{m+1} - y_m)$ is bounded, and $(y_m \cdot \mathbf{e}_i)$ is bounded for all $i \le N-1$, there exist a sequence $\delta_m \to 0$, a sequence $(a_m) \subset \mathbf{R}$ with $a_m\mathbf{e}_N$ a δ_m-almost period of h for all m, and a subsequence of (u_m) (also denoted (u_m)) with $\sup_{m\ge1} |y_m - a_m\mathbf{e}_N| < \infty$. Apply Proposition 2.1 to this latest (u_m) to obtain yet another subsequence (u_m), and $(x_m) \subset \mathbf{R}^N$, $\bar{h} \in Hull(h)$, and v satisfying (2.1)(i)-(iv). As before, it suffices to show $\bar{h}$ is a translate of h. By similar arguments to before, $(x_m - y_m)$ is bounded, so by taking a subsequence, we may assume $(x_m - a_m\mathbf{e}_N)$ is convergent, with $x_m - a_m\mathbf{e}_N \to z$. Since $a_m\mathbf{e}_N$ is a δ_m-almost period of h, $a_m\mathbf{e}_N$ is a δ_m-almost period of any translate of h as well. $\tau_{-x_m}h \to \bar{h}$ uniformly, so $\tau_{-z-a_m\mathbf{e}_N}h \to \bar{h}$ uniformly, and $\bar{h} = \tau_{-z}h$. Proposition 1.13 is proven.

Proof of Proposition 1.14

Let $d > 0$ such that $[c_0, c_0 + d)$ contains no critical values of I. Fix $l \in 0, 1, \ldots, N-2$ and assume $c_l = c_0$. Let $\epsilon \in (0, d)$. Let $\{i_1, \ldots, i_l\} \subset \{1, \ldots, N-1\}$ and $\gamma \in \Gamma[i_1, \ldots, i_l]$ with

$$\max_{\theta\in[0,1]} I(\gamma(\theta)) < c_0 + \epsilon. \tag{3.4}$$

For convenience, assume without loss of generality that $\gamma \in \Gamma[1, 2, \ldots, l]$. Let η be as in the proof of Proposition 1.13. As before, there exists $\theta \in [0,1]$ with

$$\lim_{t\to\infty} I(\eta(t, \gamma(\theta))) \in [c_0, c_0 + \epsilon). \tag{3.5}$$

Let $h^+ = \sup_{\mathbf{R}^N} h$ and let s be as in (f_3). Let $P > 0$ be large enough so that if $\Omega = \mathbf{R}^N$ or the half-space $\{x \in \mathbf{R}^N \mid x_1 > 0\}$, then $\|u\|_{L^{s+1}(\Omega)} \le P\|u\|_{W^{1,2}(\Omega)}$. By $(f_1) - (f_3)$, there exists $A > 0$ such that for all $q \in \mathbf{R}$, $|f(q)| \le |q|/(2h^+) + A|q|^s$. For $\Omega \subset \mathbf{R}^N$, define $\|u\|_\Omega = \|u\|_{W^{1,2}(\Omega)}$ and J_Ω on $W^{1,2}(\Omega)$ by $J_\Omega(u) = \|u\|_\Omega^2 - \int_\Omega hf(u)u$. Then $J_{\mathbf{R}^N}(u) = I'(u)u$. Define

$$r_0 = \frac{1}{3}\Big(\frac{1}{4h^+AP^{s+1}}\Big)^{\frac{1}{s-1}}. \tag{3.6}$$

If $\Omega = \mathbf{R}^N$ or $\{x_1 > 0\}$, and $\|u\|_\Omega \le 3r_0$, then

$$J_\Omega(u) = \|u\|_\Omega^2 - \int_\Omega hf(u)u \ge \|u\|_\Omega^2 - h^+\int_\Omega f(u)u \ge \tag{3.7}$$

$$\geq \|u\|_\Omega^2 - h^+ \int_\Omega \frac{u^2}{2h^+} + A|u|^{s+1} \geq \frac{1}{2}\|u\|_\Omega^2 - Ah^+ \int_\Omega |u|^{s+1} =$$
$$= \frac{1}{2}\|u\|_\Omega^2 - Ah^+\|u\|_{L^{s+1}(\Omega)}^{s+1} \geq \frac{1}{2}\|u\|_\Omega^2 - Ah^+P^{s+1}\|u\|_\Omega^{s+1} =$$
$$= \|u\|_\Omega^2(\frac{1}{2} - h^+AP^{s+1}\|u\|_\Omega^{s-1}) \geq$$
$$\geq \|u\|_\Omega^2(\frac{1}{2} - h^+AP^{s+1}(3r_0)^{s-1}) \geq \|u\|_\Omega^2/4.$$

The following estimates will also be useful. For any $w \in E$,

$$\|w\|^2 = 2I(w) + 2\int_{\mathbf{R}^N} hF(u) \leq 2I(w) + \frac{2}{\mu}\int_{\mathbf{R}^N} hf(w)w =$$
$$= 2I(w) + \frac{2}{\mu}(\|w\|^2 - I'(w)w),$$

so

(3.8) $$\|w\|^2 \leq \frac{2\mu I(w) - 2I'(w)w}{\mu - 2},$$

and with $h_- = \inf_{\mathbf{R}^N} h > 0$,

(3.9) $$\int_{\mathbf{R}^N} wf(w) \leq \frac{1}{h_-}\int_{\mathbf{R}^N} hwf(w) = \frac{1}{h_-}(\|w\|^2 - I'(w)w) \leq$$
$$\leq \left(\frac{\mu}{h_-}\right)\frac{2I(w) - I'(w)w}{\mu - 2}$$

and

(3.10) $$\int_{\mathbf{R}^N} F(w) \leq \frac{1}{\mu}\int_{\mathbf{R}^N} wf(w) \leq \left(\frac{1}{h_-}\right)\frac{2I(w) - I'(w)w}{\mu - 2}.$$

Let $\eta \equiv \eta(t) \equiv \eta(t, \gamma(\theta))$. Since $I'(\eta(t)) \to 0$, by (3.8) and (3.7), we may pick $t_0 > 0$ to be large enough so that for all $t \geq t_0$,

(3.11) $$|I'(\eta(t))\eta(t)| < \min(1, \epsilon, r_0^2/8) \text{ and } \|\eta(t)\| > 3r_0.$$

For $i = l+1, l+2, \ldots, N-1$, let $T_i^- < 0 < T_i^+$, with $T_i^-\mathbf{e}_i$ and $T_i^+\mathbf{e}_i$ both ϵ-almost periods of h for all i (this is possible by [S1]), and $|T_i^-|$ and T_i^+ large enough to satisfy

(3.12) $$\|\eta(t_0)\|_{\{x_i > T_i^+\}} < r_0$$

and

(3.13) $$\|\eta(t_0)\|_{\{x_i < T_i^-\}} < r_0$$

Two cases are possible:

(3.14) I. For all $t \geq t_0$ and $i = l+1, l+2, \ldots, N-1$,

$$\|\eta(t)\|_{\{x_i < T_i^+\}} > r_0 \text{ and } \|\eta(t)\|_{\{x_i > T_i^-\}} > r_0.$$

II. Case I does not hold.

In Case I, let $u_m = \eta(m)$. As in the proof of Proposition 1.13, $I'(u_m) \to 0$ as $m \to \infty$ and $\|u_{m+1} - u_m\| \to 0$. As before, there exist a sequence $(x_m) \subset \mathbf{R}^N$ and $\rho > 0$ such that $(x_{m+1} - x_m)$ is bounded and $\|u_m\|_{W^{1,2}(B_1(x_m))} > \rho$ for large m. Assume without loss of generality that $c_0 > d > \epsilon$. By Proposition 2.1, and the fact that $u_m \in E[1, 2, \ldots, l]$, $(x_m \cdot \mathbf{e}_i)$ is bounded in m for $i = 1, 2, \ldots, N-1$. The argument from the proof of Proposition 1.13 shows that I has a critical point with critical level in the interval $[c_0, c_0 + \epsilon)$, contradicting our assumption that $[c_0, c_0 + d)$ contains no critical values of I. Thus Case II holds.

For Case II, assume without loss of generality that Case I fails for $\{x_{l+1} < T_{l+1}^+\}$. By (3.11), $\|\eta(t)\| > 3r_0$ for all $t \geq t_0$, so (3.12) implies $\|\eta(t_0)\|_{\{x_{l+1} < T_{l+1}^+\}} \geq 2r_0$, and we can define

$$t_2 = \min\{t > t_0 \mid \|\eta(t)\|_{\{x_{l+1} < T_{l+1}^+\}} = r_0\} > t_0. \tag{3.15}$$

Now $\|\eta(t_2)\|_{\{x_{l+1} > T_{l+1}^+\}} \geq 2r_0$, so by (3.12), we can define

$$t_1 = \max\{t < t_2 \mid \|\eta(t)\|_{\{x_{l+1} > T_{l+1}^+\}} = r_0\} \in (t_0, t_2). \tag{3.16}$$

For all $t \in (t_1, t_2)$,

$$\|\eta(t)\|_{\{x_{l+1} < T_{l+1}^+\}} > r_0 \text{ and } \|\eta(t)\|_{\{x_{l+1} > T_{l+1}^+\}} > r_0. \tag{3.17}$$

By (3.7) and (3.16),

$$J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t_1)) \geq \|\eta(t_1)\|^2_{\{x_{l+1} > T_{l+1}^+\}}/4 = r_0^2/4 > 0. \tag{3.18}$$

Likewise, by (3.7) and (3.15),

$$J_{\{x_{l+1} < T_{l+1}^+\}}(\eta(t_2)) \geq r_0^2/4, \tag{3.19}$$

so by (3.19) and (3.11),

$$J_{\{x_{l+1}>T^+_{l+1}\}}(\eta(t_2)) = J_{\mathbf{R}^N}(\eta(t_2)) - J_{\{x_{l+1}<T^+_{l+1}\}}(\eta(t_2)) \le \le I'(u)u - r_0^2/4 < r_0^2/8 - r_0^2/4 < 0. \tag{3.20}$$

By (3.18) and (3.20), there exists $t^* \in (t_1, t_2)$ with $J_{\{x_{l+1}>T^+_{l+1}\}}(\eta(t^*)) = 0$, so by (3.11),

$$\max(|J_{\{x_{l+1}>T^+_{l+1}\}}(\eta(t^*))|\,,\,|J_{\{x_{l+1}<T^+_{l+1}\}}(\eta(t^*))|) < \epsilon. \tag{3.21}$$

Define

$$z = \tau_{-T^+_{l+1}\mathbf{e}_{l+1}}\eta(t^*). \tag{3.22}$$

Since $d < c_0$, (3.8)-(3.10) imply that for all $t \ge t_0$,

$$\max\left(|I'(\eta(t))\eta(t)|, \int_{\mathbf{R}^N} F(\eta(t)), \int_{\mathbf{R}^N} f(\eta(t))\eta(t)\right) \le \le 1 + \frac{\mu}{h_-}\left(\frac{4c_0+1}{\mu-2}\right) \equiv K(f,h). \tag{3.23}$$

Thus

$$\begin{aligned} |I(z) - I(\eta(t^*))| &= |\int_{\mathbf{R}^N} hF(z) - \int_{\mathbf{R}^N} hF(\eta(t^*))| = \\ &= \left|\int_{\mathbf{R}^N} h(x)F((\eta(t^*)(x - T^+_{l+1}\mathbf{e}_{l+1}))\,dx - \int_{\mathbf{R}^N} h(x)F(\eta(t^*)(x))\,dx\right| = \\ &= |\int_{\mathbf{R}^N} (h(y + T^+_{l+1}\mathbf{e}_{l+1}) - h(y))F(\eta(t^*)(y))\,dy| \le \\ &\le \epsilon \int_{\mathbf{R}^N} F(\eta(t^*)) \le \epsilon K(f,h). \end{aligned} \tag{3.24}$$

Similarly,

$$\begin{aligned} &|J_{\{x_{l+1}>0\}}(z) - J_{\{x_{l+1}>T^+_{l+1}\}}(\eta(t^*))| = \\ &= |\int_{x_{l+1}>0} hf(z)z - \int_{x_{l+1}>T^+_{l+1}} hf(\eta(t^*))\eta(t^*)| = \\ &= |\int_{x_{l+1}>T^+_{l+1}} h(x - T^+_{l+1}\mathbf{e}_{l+1})f(z(x - T^+_{l+1}\mathbf{e}_{l+1}))z(x - T^+_{l+1}\mathbf{e}_{l+1}) - \\ &\qquad h(x)f(\eta(t^*)(x))\eta(t^*)(x)\,dx\Big| = \\ &= |\int_{x_{l+1}>T^+_{l+1}} (h(x - T^+_{l+1}\mathbf{e}_{l+1}) - h(x))f(\eta(t^*)(x))\eta(t^*)(x)\,dx| \le \\ &\le \epsilon \int_{x_{l+1}>T^+_{l+1}} f(\eta(t^*)(x))\eta(t^*)(x)\,dx \le \epsilon \int_{\mathbf{R}^N} f(\eta(t^*))\eta(t^*) \le \epsilon K(f,h) \end{aligned} \tag{3.25}$$

and

$$|J_{\{x_{l+1}<0\}}(z) - J_{\{x_{l+1}<T^+_{l+1}\}}(\eta(t^*))| \le K(f,h)\epsilon. \tag{3.26}$$

For $\Omega \subset \mathbf{R}^N$, define I_Ω by $I_\Omega(w) = \frac{1}{2}\|w\|^2_\Omega - \int_\Omega hF(w)$. By (3.24),

$$I_{\{x_{l+1}<0\}}(z) + I_{\{x_{l+1}>0\}}(z) = I(z) < I(\eta(t^*)) + \epsilon K(f,h) <$$
$$< c_0 + \epsilon(1 + K(f,h)),$$

so

$$\min(I_{\{x_{l+1}<0\}}(z), I_{\{x_{l+1}>0\}}(z)) < \frac{1}{2}(c_0 + \epsilon(1 + K(f,h))). \tag{3.27}$$

Assume without loss of generality that $I_{\{x_{l+1}>0\}}(z) < \frac{1}{2}(c_0 + \epsilon(1 + K(f,h)))$, and define $\hat{z}$ by reflecting the $\{x_{l+1} > 0\}$-half of z over the x_{l+1}-axis. That is,

$$\hat{z}(x) = \begin{cases} z(x); & x_{l+1} \ge 0 \\ z(R_{l+1}x); & x_{l+1} < 0, \end{cases} \tag{3.28}$$

where the reflection operator R_i is as in (1.6). Then $\hat{z} \in E[1, 2, \ldots, l+1]$.

Now $\|\hat{z}\| \le 2\|\eta(t^*)\| < 2K(f,h)$ by (3.23). Also,

$$\|\hat{z}\| \ge \|\hat{z}\|_{\{x_{l+1}>0\}} = \|\eta(t^*)\|_{\{x_{l+1}>T^+_{l+1}\}} > r_0 \tag{3.29}$$

by (3.17),

$$|I'(\hat{z})\hat{z}| = |J_{\mathbf{R}^N}(\hat{z})| = 2|J_{\{x_{l+1}>0\}}(\hat{z})| = 2|J_{\{x_{l+1}>0\}}(z)| \le \tag{3.30}$$
$$\le 2(|J_{\{x_{l+1}>T^+_{l+1}\}}(\eta(t^*))| + \epsilon K(f,h)) < 2\epsilon(1 + K(f,h))$$

by (3.21) and (3.25). Also, $I(\hat{z}) = 2I_{\{x_{l+1}>0\}}(z) < c_0 + \epsilon(1 + K(f,h))$ by (3.27).

Letting $\epsilon_m \to 0$ and constructing $\hat{z}_m$ in this manner, we obtain a bounded sequence $(\hat{z}_m) \subset E_{l+1}$ as in (2.12) with $\|\hat{z}_m\|$ bounded away from zero, $I'(\hat{z}_m)\hat{z}_m \to 0$, and $\limsup_{m\to\infty} I(z) \le c_0$. Therefore, by (2.12), $c_{l+1} \le c_0$. Thus $c_{l+1} = c_0$, and Proposition 1.14 is proven.

Positive Solutions

By the argument in the Introduction, for any $\epsilon > 0$, there is a critical value of I in the interval $[c_0, c_0 + \epsilon)$. To complete the proof of Theorem 1.2, let $\epsilon_m \to 0$, and let u_m

be a critical point of I with $I(u_m) \in [c_0, c_0 + \epsilon_m)$. By Proposition 2.1, there exists $\bar{h}$ in the hull of h, a sequence $(x_m) \subset \mathbf{R}^N$, and $v \in E$ such that $I[\bar{h}]'(v) = 0$, $I[\bar{h}](v) = c_0$, and $\|u_m - \tau_{x_m} v\| \to 0$ as $m \to \infty$. Since c_0 is the minimax value associated with $I[\bar{h}]$, arguments of [CR1], employing (f_5), show that v does not change sign. Without loss of generality, v is nonnegative. Since $\tau_{-x_m} u_m \to v$ in $W^{1,2}(\mathbf{R}^N)$, $\tau_{-x_m} u_m \to v$ uniformly. Then a standard maximum principle argument shows that for large m, u_m is a positive function.

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