# Traveling Waves for the BBM equation with viscosity term

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**Abstract**

The Benjamin-Bona-Mahony equation is a popular alternative to the Korteweg-de Vries equation, for modeling unidirectional propagation of long waves in systems that manifest nonlinear and dispersive effects. In this paper, it is shown that when a viscosity term is added to the BBM equation, there exist bounded traveling wave solutions.

## 1

The Benjamin-Bona-Mahony equation or regularized long-wave equation:

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.1}$$

is a popular alternative to the Korteweg-de Vries equation, for modeling unidirectional propagation of long waves in systems that manifest nonlinear and dispersive effects. It was introduced by Peregrine ([4]). In [1], the equation is recast as an integral equation, and solutions, global in time, are found for initial data in $W^{1,2}(\mathbb{R})$. In [2], global solutions are found for initial data in $H^s(\mathbb{R})$, $s \geq 0$. Exact analytical traveling-wave solutions to the BBM equation are found in [3].

Consider the BBM or regularized long-wave equation with an added viscosity term:

$$u_t + u_x + uu_x - u_{xxt} = \nu u_{xx} \tag{1.2}$$

with $\nu > 0$. The equation has bounded traveling-wave solutions. We will prove the following:

**Theorem 1.1.** *Let $c \neq 0$ and $k > 0$. (1.2) has a solution of the form $u(x,t) = f(x - ct)$ with $f(\xi) \to (c-1) + k$ as $\xi \to -\infty$ and $f(\xi) \to (c-1) - k$ as $\xi \to \infty$.*

Proof: substituting the *ansatz* $u(x,t) = f(x - ct)$ into (1.2) yields

$$(1-c)f' + ff' - cf''' = \nu f''. \tag{1.3}$$

Integrating,

$$(1-c)f + \frac{1}{2}f^2 - cf'' = \nu f' + A \tag{1.4}$$

for some constant $A$. With $f_1 = f$ and $f_2 = f'$, (1.4) is equivalent to the dynamical system

$$\begin{aligned} f_1' &= f_2 \\ f_2' &= \frac{1}{2c}f_1^2 + (\frac{1-c}{c})f_1 - \frac{\nu}{c}f_2 - \frac{A}{c}. \end{aligned} \tag{1.5}$$

Assuming $(c-1)^2 + 2A > 0$, (1.5) has two fixed points, $\begin{bmatrix} u^{\pm} \\ 0 \end{bmatrix}$, where

$$u^{\pm} = (c-1) \pm \sqrt{(c-1)^2 + 2A}. \tag{1.6}$$

$A$ is chosen so $\sqrt{(c-1)^2 + 2A} = k$, where $k$ is from the statement of Theorem 1.1.

Define

$$P_3(x) = \frac{1}{6}x^3 + \frac{1}{2}(1-c)x^2 - Ax \tag{1.7}$$

and

$$\mathcal{L}(\mathbf{f}) = P_3(f_1) - \frac{1}{2}cf_2^2 = \frac{1}{6}f_1^3 + \frac{1}{2}(1-c)f_1^2 - Af_1 - \frac{1}{2}cf_2^2. \tag{1.8}$$

Multiplying (1.4) by the integrating factor $f'$, we obtain

$$\frac{d}{d\xi}\mathcal{L}(\mathbf{f}(\xi)) = \nu f_2(\xi)^2 \tag{1.9}$$

for any solution $\mathbf{f}$ of (1.5).

There are two cases, $c > 0$ and $c < 0$. First, assume that $c > 0$. Then the linearized system to (1.5) at the fixed point $\begin{bmatrix} u^{+} \\ 0 \end{bmatrix}$ is

$$\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{(c-1)^2+2A}}{c} & -\frac{\nu}{c} \end{bmatrix} \begin{bmatrix} f_1 - u^+ \\ f_2 \end{bmatrix} \equiv M \begin{bmatrix} f_1 - u^+ \\ f_2 \end{bmatrix}. \tag{1.10}$$

$det(M) < 0$, so $\begin{bmatrix} u^{+} \\ 0 \end{bmatrix}$ is a saddle point of (1.5). $M$ has eigenvalues

$$\lambda^{\pm} = \frac{-\nu \pm \sqrt{\nu^2 + 4c\sqrt{(c-1)^2 + 2A}}}{2c} \tag{1.11}$$

with $\lambda^- < 0 < \lambda^+$. The unstable manifold of (1.5) at $\begin{bmatrix} u^{+} \\ 0 \end{bmatrix}$ has slope $\lambda^+$.

$\mathcal{L}$ is a Lyapunov function for the flow generated by (1.5). Consider the set $C \subset \mathbb{R}^2$ defined by

$$C = \{\mathbf{f} \in \mathbb{R}^2 \mid \mathcal{L}(\mathbf{f}) \geq P_3(u^+) = \mathcal{L}(\begin{bmatrix} u^{+} \\ 0 \end{bmatrix})\}. \tag{1.12}$$

$C$ can also be defined by

$$C = \{\mathbf{f} \in \mathbb{R}^2 \mid \frac{1}{2}cf_2^2 \leq P_3(f_1) - P_3(u^+)\}. \tag{1.13}$$

The mapping $f_1 \mapsto P_3(f_1) - P_3(u^+)$ is a cubic function of $f_1$ that is increasing on $(-\infty, u^-)$, has a positive local maximum at $u^-$, decreases on $(u^-, u^+)$, equals zero at $f_1 = u^+$, and increases to infinity on $(u^+, \infty)$. Therefore the interior of $C$ has two components, a bounded component $\Omega$ to the left of $\{f_1 = u^+\}$ containing the fixed point $\begin{bmatrix} u^- \\ 0 \end{bmatrix}$, and an unbounded component to the right of $\{f_1 = u^+\}$. $\Omega$ is a forward-invariant set for (1.5), because $\mathcal{L} = \mathcal{L}(\begin{bmatrix} u^+ \\ 0 \end{bmatrix})$ on $\partial\Omega$ and $\mathcal{L}(\mathbf{f}) > \mathcal{L}(\begin{bmatrix} u^+ \\ 0 \end{bmatrix})$ for $\mathbf{f} \in \Omega$, and (1.9).

The boundary of $C$ has the form

$$\partial C = \{\mathbf{f} \mid f_2^2 = \frac{2}{c}(P_3(f_1) - P_3(u^+))\}. \tag{1.14}$$

Now

$$\frac{d^2}{df_1^2}(\frac{2}{c}(P_3(f_1) - P_3(u^+)))\Big|_{f_1=u^+} = \frac{2\sqrt{(c-1)^2 + 2A}}{c}, \tag{1.15}$$

so the boundary of $C$ (and the boundary of $\Omega$) has slope $\pm\sqrt{\frac{1}{c}} \cdot \sqrt[4]{(c-1)^2 + 2A}$ at $\begin{bmatrix} u^+ \\ 0 \end{bmatrix}$. It is a simple exercise to verify that

$$\lambda^+ = \frac{-\nu + \sqrt{\nu^2 + 4c\sqrt{(c-1)^2 + 2A}}}{2c} < \sqrt{\frac{1}{c}} \cdot \sqrt[4]{(c-1)^2 + 2A}. \tag{1.16}$$

Therefore the unstable manifold at $\begin{bmatrix} u^+ \\ 0 \end{bmatrix}$ points into $\Omega$.

Let $\mathbf{u}$ be a solution of (1.5) satisfying $\mathbf{u}(\xi) \to \begin{bmatrix} u^+ \\ 0 \end{bmatrix}$ as $\xi \to -\infty$ and $\mathbf{u}(\xi) \in \Omega$ for all real $\xi$. It remains to be shown that $\mathbf{u}(\xi) \to \begin{bmatrix} u^- \\ 0 \end{bmatrix}$ as $\xi \to \infty$. We will show that the $\omega$-limit set of $\mathbf{u}$ is $\{\begin{bmatrix} u^- \\ 0 \end{bmatrix}\}$. Since $\bar{\Omega}$ is compact, the $\omega$-limit set of $\mathbf{u}$ is nonempty. Let $(\xi_m)_{m\geq 1}$ with $\xi_m \to \infty$ as $m \to \infty$ and $u(\xi_m) \to \mathbf{x} \in \Omega$ as $m \to \infty$. Suppose $\mathbf{x} \neq \begin{bmatrix} u^- \\ 0 \end{bmatrix}$. Let $\mathbf{z}$ be the solution of the initial value problem (1.5), $\mathbf{z}(0) = \mathbf{x}$. By (1.5), there exists $\epsilon > 0$ with $z_2(\xi) \neq 0$ for $0 < \xi < \epsilon$, so by (1.9), $\mathcal{L}(\mathbf{z}(\epsilon)) > \mathcal{L}(\mathbf{x})$. As $m \to \infty$, $\mathbf{u}(\xi_m) \to \mathbf{x}$ and $\mathbf{u}(\xi_m + \epsilon) \to \mathbf{z}(\epsilon)$, so there exist $m < n$ with $\xi_m + \epsilon < \xi_n$ and $\mathcal{L}(\xi_m + \epsilon) > \mathcal{L}(\xi_n)$. This contradicts (1.9). Thus the $\omega$-limit set of $\mathbf{u}$ is $\{\begin{bmatrix} u^- \\ 0 \end{bmatrix}\}$, and $\mathbf{u}(\xi) \to \begin{bmatrix} u^- \\ 0 \end{bmatrix}$ as $\xi \to \infty$. Theorem 1.1 is proven for the $c > 0$ case.

The $c < 0$ case is similar. The linearized system to (1.5) at $\begin{bmatrix} u^- \\ 0 \end{bmatrix}$ is

$$\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{(c-1)^2+2A}}{c} & -\frac{\nu}{c} \end{bmatrix} \begin{bmatrix} f_1 - u^- \\ f_2 \end{bmatrix}. \tag{1.17}$$

$\begin{bmatrix} u^- \\ 0 \end{bmatrix}$ is a saddle point, whose stable manifold has slope

$$\lambda^- = \frac{-\nu + \sqrt{\nu^2 - 4c\sqrt{(c-1)^2 + 2A}}}{2c} < 0. \tag{1.18}$$

Define $C \subset \mathbb{R}^2$ by

$$C = \{\mathbf{f} \in \mathbb{R}^2 \mid \mathcal{L}(\mathbf{f}) \le P_3(u^-) = \mathcal{L}(\begin{bmatrix} u^- \\ 0 \end{bmatrix})\}. \tag{1.19}$$

$C$ can also be defined by

$$C = \{\mathbf{f} \in \mathbb{R}^2 \mid \frac{1}{2}cf_2^2 \ge P_3(f_1) - P_3(u^-)\}. \tag{1.20}$$

The interior of $C$ has two components, a bounded component $\Omega$ to the left of $\{f_1 = u^-\}$, and an unbounded component to the right of $\{f_1 = u^-\}$. For $\mathbf{x} \in \Omega$, $\mathcal{L}(\mathbf{x}) < \mathcal{L}(\begin{bmatrix} u^- \\ 0 \end{bmatrix})$. Thus $\Omega$ is a backward-invariant set for the flow induced by (1.5); if $\mathbf{w}$ solves (1.5) and $\mathbf{w}(0) \in \Omega$, then $\mathbf{w}(\xi) \in \Omega$ for all $\xi < 0$.

Like before, the slope of $\partial\Omega$ at $\begin{bmatrix} u^- \\ 0 \end{bmatrix}$ is $\pm\sqrt{\frac{1}{|c|}} \cdot \sqrt[4]{(c-1)^2 + 2A}$. It is easy to verify that

$$|\lambda^-| = \frac{-\nu + \sqrt{\nu^2 - 4c\sqrt{(c-1)^2 + 2A}}}{2|c|} < \sqrt[4]{\frac{1}{|c|}} \cdot \sqrt{(c-1)^2 + 2A}, \tag{1.21}$$

so the stable manifold of $\begin{bmatrix} u^- \\ 0 \end{bmatrix}$ points into $\Omega$.

Let $\mathbf{u}$ be a solution of (1.5) with $\mathbf{u}(\xi) \to \begin{bmatrix} u^- \\ 0 \end{bmatrix}$ as $\xi \to \infty$ and $\mathbf{u}(\xi) \in \Omega$ for all real $\xi$. By a similar argument to before, the $\alpha$-limit set of $\begin{bmatrix} u^- \\ 0 \end{bmatrix}$ consists of one point, $\begin{bmatrix} u^+ \\ 0 \end{bmatrix}$. Thus $\mathbf{u}(\xi) \to \begin{bmatrix} u^+ \\ 0 \end{bmatrix}$ as $\xi \to -\infty$. Theorem 1.1 is proven for the case $c < 0$.

□

## References


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