Interacting Near-Solutions of a Hamiltonian System

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ABSTRACT: A Hamiltonian system with a superquadratic potential is examined. The system is asymptotic to an autonomous system. The difference between the Hamiltonian system and the "problem at infinity," the autonomous system, may be large, but decays exponientially. The existence of a nontrivial solution homoclinic to zero is proven. Many results of this type rely on a monotonicity condition on the nonlinearity, not assumed here, which makes the problem resemble in some sense the special case of homogeneous (power) nonlinearity.

The proof employs variational, minimax arguments. In some similar results requiring the monotonicity condition, solutions inhabit a manifold homeomorphic to the unit sphere in a the appropriate Hilbert space of functions. An important part of the proof here is the construction of a similar set, using only the mountain-pass geometry of the energy functional. Another important element is the interaction between functions resembling widely separated solutions of the autonomous problem.

KEY WORDS: *Mountain Pass Theorem, variational methods, concentration-compactness, Nehari manifold, homoclinic solutions*
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1. Introduction

This paper is inspired by a result of Bahri and Li ([BL]). The authors studied an elliptic partial differential equation of the form $-\Delta u + u = b(x)u^p$, $x \in \mathbf{R}^N$, with

$b(x) \to b_\infty > 0$ as $|x| \to \infty$ and the negative part of $b(x) - b_\infty$ decaying exponentially to zero as $|x| \to \infty$. They showed that a positive solution v exists with $v(x) \to 0$ as $|x| \to \infty$.

The proof employed a minimax argument. The "problem at infinity," $-\Delta u + u = b_\infty u^p$, has a unique (modulo translation) positive solution w with $w(x) \to 0$ as $|x| \to \infty$. The minimax argument uses sums of translates of w, and exploits how the "tails" of these translates interact. A similar concept is found in [WX].

A natural problem is to generalize the result of [BL] to the case of a nohomogeneous nonlinearity. A step in this direction was taken by Adachi ([A]), in which u^p is replaced by a more general superlinear function of u. The coefficient function $b(x)$ is assume to possess symmetry, however.

[S] examines an $N = 1$ version of the problem, $-u'' + u = b(t)f(u)$. Like in the PDE problem, $f(u)$ is a superlinear function of u, $b(t) \to b_\infty > 0$ as $|t| \to \infty$, and the negative part of $b(t) - b_\infty$ decays to zero suitably rapidly as $|t| \to \infty$. It is proven that this Hamiltonian system has a positive solution homoclinic to zero. The proof involves translates of the unique positive, even solution of the problem at infinity, $-u'' + u = b_\infty f(u)$, and exploits the interaction of their "tails." The function f satisfies the Ambrosetti-Rabinowitz condition (described later), which ensures that $f(u)$ behaves like a superlinear power of u. Another critical assumption is that $f(q)/q$ is a nondecreasing function of q for positive q. This popular assumption, also found in [AM],[CMN], [STT] and elsewhere, has many helpful implications.

The present paper proves a result similar to [S], while dispensing with the assumption that $f(q)/q$ be nondecreasing. We examine a Hamiltonian system of the form

$$-u'' + u = W'(t, u), \tag{1.1}$$

and prove the following:

THEOREM 1.2 *Let $N \in \mathbf{N}^+$, and let V and W satisfy*

(V_1) $V \in C^{1,1}(\mathbf{R}^+, \mathbf{R}^+)$

(V_2) $V(0) = 0$

(V_3) *There exists* $\mu > 2$ *such that* $V'(q)q \geq \mu V(q) > 0$ *for all* $q > 0$

(W_1) $W \in C^{1,1}(\mathbf{R} \times \mathbf{R}^N, \mathbf{R}^N)$

(W_2) $W(t, 0) = 0$ *for all* $t \in \mathbf{R}$

(W_3) $W'(t, u)u \geq \mu W(t, u) > 0$ *for all* $t \in \mathbf{R}, u \in \mathbf{R}^N \setminus \{0\}$, *where* μ *is from* (V_3) *and* $W'(t, u) = < \frac{\partial}{\partial u_1} W(t, u), \frac{\partial}{\partial u_2} W(t, u), \ldots, \frac{\partial}{\partial u_N} W(t, u) >$.

(W_4) $(W(t, u) - V(|u|))/V(|u|) \to 0$ *as* $|t| \to \infty$, *uniformly in* $u \in \mathbf{R}^N \setminus \{0\}$.

(W_5) $W'(t, u)u/|u|^2 \to 0$ *as* $|u| \to 0$, *uniformly in* $t \in \mathbf{R}$.

(W_6) *There exist* $\delta > 2\mu/(\mu - 2)$ *and* $A > 0$ *with* $W(t, u) - V(u) \geq -AV(|u|)e^{-\delta|t|}$ *for all* $t \in \mathbf{R}$, $u \in \mathbf{R}^N$, *or* $W(t, u) - V(u) \geq -A|u|^\mu e^{-\delta|t|}$ *for all* $t \in \mathbf{R}$, $u \in \mathbf{R}^N$.

Then (1.1) has a nontrivial solution v *homoclinic to zero, with* $I(v) \in (0, 2c_0)$, *where* c_0 *is the mountain pass value associated with the function* J *(see (1.7)-(1.9)).*

The variational framework

Define the C^2 functional $I : W^{1,2}(\mathbf{R}, \mathbf{R}^N) \to \mathbf{R}$ by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} W(t, u(t))\, dt, \tag{1.3}$$

where $\| \ \|$ is the standard norm, $\|u\|^2 = \int_{-\infty}^{\infty} |u'(t)|^2 + u(t)^2\, dt$. Critical points of I correspond exactly to solutions of (1.1) homoclinic to zero. The conditions (V_3) and (W_3) ensure that V and W are "superquadratic" functions, with, for example, $V(q)/q^2 \to 0$ as $q \to 0$ and $V(q)/q^2 \to \infty$ as $q \to \infty$. Thus $(W_1) - (W_3)$ imply I has "mountain pass" geometry. That is, $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $\|u\| \to 0$, and $I(u) < 0$ for some $u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N)$. Therefore the set of mountain pass curves

$$\Gamma = \{\gamma \in C([0, 1], W^{1,2}(\mathbf{R}, \mathbf{R}^N)) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\} \tag{1.4}$$

is nonempty, and the "mountain pass" value c defined by

$$c = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I(\gamma(\theta)) \tag{1.5}$$

is positive.

We will show that the scalar differential equation

$$-u'' + u = V'(u) \tag{1.6}$$

can be regarded as the "problem at infinity" for (1.1). This equation has a unique (modulo translation) positive solution ω. ω is even, increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$.

Extend V to the negative reals by making V even, that is, $V(-q) = V(q)$ for all $q \in \mathbf{R}$. Then the functional $J \in C^2(W^{1,2}(\mathbf{R}, \mathbf{R}), \mathbf{R})$ corresponding to (1.6) is

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{-\infty}^{\infty} V(u(t))\, dt, \tag{1.7}$$

where $\|u\|^2 = \int_{-\infty}^{\infty} u'(t)^2 + u(t)^2\, dt$. Like I, J has mountain-pass geometry, with

$$\Gamma_0 = \{\gamma \in C([0,1], W^{1,2}(\mathbf{R}, \mathbf{R})) \mid \gamma(0) = 0,\ J(\gamma(1)) < 0\} \tag{1.8}$$

nonempty, and the mountain pass value c_0 defined by

$$c_0 = \inf_{\gamma \in \Gamma_0} \max_{\theta \in [0,1]} I(\gamma(\theta)) \tag{1.9}$$

positive.

The Missing Monotonicity Assumption

An important feature of Theorem 1.2 is a condition that is *not* assumed. We do not assume that

$$W(t, su)/s^2 \text{ is a nondecreasing function of } s \tag{1.10}$$
$$\text{for positive } s \text{ and all } t \in \mathbf{R} \text{ and } u \in \mathbf{R}^N \setminus \{0\},$$

or that

$$V(q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q > 0. \tag{1.11}$$

The implications of such an assumption have been studied by Nehari. The first assumption would imply that for any $u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N)$, the mapping $s \mapsto I(su)$ starts at 0 when $s = 0$, increases to a positive maximum, then decreases to $-\infty$ (see [CR]). Then we could define the "Nehari manifold"

$$\mathcal{S} = \{u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N) \mid I'(u)u = 0,\ u \neq 0\}. \tag{1.12}$$

$\mathcal{S}$ would be a codimension-one manifold of $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$, homeomorphic to the unit sphere in $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$ via radial projection. All nonzero critical points of I would be in $\mathcal{S}$. The minimax value c could be described more simply as

$$c = \inf_{u \in \mathcal{S}} I(u). \tag{1.13}$$

Also, for any $u \in \mathbf{R}^N \setminus \{0\}$, the function γ, defined by $\gamma(\theta) = T\theta u$ for some suitable scaling factor T, would belong to Γ. Therefore we could work with nonzero functions in $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$ instead of curves in Γ. [S] takes advantage of these facts.

In this paper (1.10) is not assumed, so $\mathcal{S}$ may not have these properties. Instead, the mountain pass geometry of I is used to constuct a set with similar properties to $\mathcal{S}$. The set is defined as follows: let ∇I denote the gradient of I. That is, $(\nabla I(u), w) = I'(u)w$ for all $u, w \in W^{1,2}(\mathbf{R}, \mathbf{R}^N)$. Here, $(\cdot, \cdot)$ is the usual inner product defined by $(u, w) = \int_{-\infty}^{\infty} u'(t) \cdot v'(t) + u(t)v(t)\, dt$. $\nabla I(u)$ exists because of the Riesz Representation Theorem, and ∇I is a C^1 vector field. Let $\varphi : W^{1,2}(\mathbf{R}, \mathbf{R}^N) \to [0, 1]$ be a suitable locally Lipschitz cutoff function, and let η be the solution of the initial value problem

$$\frac{d\eta}{ds} = -\varphi(\eta)\nabla(\eta),\ \ \eta(0, u) = u. \tag{1.14}$$

$\varphi(\eta)$ is introduced and defined so that η is defined on $\mathbf{R}^+ \times W^{1,2}(\mathbf{R}, \mathbf{R}^N)$. Let $\mathcal{B}$ be the basin of attraction of 0 under the flow η, that is,

$$\mathcal{B} = \{u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N) \mid \eta(s, u) \to 0 \text{ as } s \to \infty\}. \tag{1.15}$$

$\mathcal{B}$ is a connected, nonempty open neighborhood of 0. Let $\partial\mathcal{B}$ denote the topological boundary of $\mathcal{B}$, under the usual metric on $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$. As with $\mathcal{S}$, any mountain pass curve $\gamma \in \Gamma$ intersects $\partial\mathcal{B}$ at least once. Also, $\partial\mathcal{B}$ is forward-η-invariant. That is, for any $u \in \partial\mathcal{B}$ and $s > 0$, $\eta(s, u) \in \partial\mathcal{B}$.

Sketch of the Proof

To prove Theorem 1.2, we assume that I has no critical point v with $0 < I(v) \leq c_0$, then show that this implies I has a critical value in the interval $(c_0, 2c_0)$. To do this, we construct a minimax class with a minimax value m strictly between c_0 and $2c_0$. There then exists a Palais-Smale sequence $(u_n) \subset W^{1,2}(\mathbf{R}, \mathbf{R}^N)$ with

$I'(u_n) \to 0$ and $I(u_n) \to m$ as $n \to \infty$. By a concentration-compactness argument, (u_n) converges along a subsequence to v, a critical point of I with $c_0 < I(v) < 2c_0$, proving Theorem 1.2.

The minimax class is constructed as follows (more details are in Sections 2-4). Construct $\gamma \in \Gamma_0$ satisfying $\max_{\theta\in[0,1]} J(\gamma(\theta)) = c_0$ and some other helpful properties. Define the translation operator τ as follows: for a function u over the reals and $a \in \mathbf{R}$, let $\tau_a u$ be u shifted a units to the right, that is, $\tau_a u(t) = u(t-a)$ for all $t \in \mathbf{R}$. Let $R_1 > 0$ be a suitably large constant, and define $\mathcal{G}$ to be a family of functions from the unit square to $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$: let $\mathbf{e}_1 = <1, 0, 0, \ldots, 0> \in \mathbf{R}^N$, and define

$$\begin{aligned}\mathcal{G} = \{G \in C([0,1]^2, W^{1,2}(\mathbf{R}, \mathbf{R}^N)) \mid \text{ for all } x, y \in [0,1], \\ G(x,0) = 0,\ I(G(x,1)) < 0, \\ G(0,y) = \tau_{-R_1}\gamma(y)\mathbf{e}_1,\ G(1,y) = \tau_{R_1}\gamma(y)\mathbf{e}_1\}.\end{aligned} \tag{1.16}$$

Then define

$$m = \inf_{G\in\mathcal{G}} \max_{(x,y)\in[0,1]^2} I(G(x,y)). \tag{1.17}$$

We will prove that, assuming I has no critical values in $(0, 2c_0]$, then $c_0 < m < 2c_0$, and there exists a Palais-Smale sequence (u_n) with $I'(u_n) \to 0$ and $I(u_n) \to m$, which converges along a subsequence. Therefore I has a critical value between c_0 and $2c_0$.

To prove $m < 2c_0$, we must construct a $G_0 \in \mathcal{G}$ with $I(G_0(x,y)) < 2c_0$ for all $(x,y) \in [0,1]^2$. Such a G_0 is given by

$$G_0(x,y) = \max(\tau_{-R_1}\gamma((1-x)y), \tau_{R_1}\gamma(xy))\mathbf{e}_1. \tag{1.18}$$

Proving $I(G_0(x,y)) < 2c_0$ is most difficult to prove if $J(\gamma((1-x)y))$ and $J(\gamma(xy))$ are both close to c_0. By the construction of γ, $\gamma((1-x)y)$ and $\gamma(xy)$ both have "tails" that are translates of the left or right half of ω. That is, for some large M, there exists $\bar{t} \geq 0$ with $\gamma(xy)(t) = \omega(|t| - M + \bar{t})$ for all $|t| > M$ (and similarly for $\gamma((1-x)y)$). $\omega(t)$ decays exponentially to zero as $|t| \to \infty$. γ is constructed so that $\gamma(\theta)(t)$ decays to zero exponentially as $|t| \to \infty$ for *all* $\theta \in [0,1]$, uniformly in θ. As the title of this paper suggests, the interaction of the exponentially decaying "tails" of $\gamma((1-x)y)$

and $\gamma(xy)$, along with the exponential decay of the negative part of $W(t,u)-V(|u|)$, imply that $I(G_0(x,y)) = I(\max(t_{-R_1}\gamma((1-x)y), t_{R_1}\gamma(xy))\mathbf{e}_1) < 2c_0$.

The paper is organized as follows: in Section 2 a suitable mountain pass curve γ is constructed. Section 3 proves some properties of $\partial\mathcal{B}$. Section 4 contains the minimax argument, proving that $\mathcal{G}$, m, and G_0 have the properties claimed in this Introduction.

2. A Mountain-Pass Curve for the Scalar Equation

By [JT], there exists $\gamma_0 \in \Gamma_0$ with $\max_{\theta\in[0,1]} J(\gamma_0(\theta)) = c_0$. This is proven under weaker assumptions than here, and is not obvious from the definition of c_0. Instead of using this result, we will construct such a γ directly, using an argument similar to that in [C]. We will prove

PROPOSITION 2.1 *There exist $M = M(V)$ and $\gamma \in \Gamma_0$ such that for all $\theta \in [0,1]$ and $t \in \mathbf{R}$,*

(i) $J(\gamma(\theta)) \leq c_0$

(ii) $\theta \geq \dfrac{1}{2} \Rightarrow J(\gamma(\theta)) < -2c_0$

(iii) $\gamma(\theta)(t) > 0$ *if* $\theta > 0$

(iv) $\gamma(\theta)(t) = \gamma(\theta)(-t)$

(v) $\theta_1 < \theta_2 \Rightarrow \gamma(\theta_1)(t) \leq \gamma(\theta)(t)$

(vi) $0 \leq t_1 \leq t_2 \Rightarrow \gamma(\theta)(t_1) \geq \gamma(\theta)(t_2)$

(vii) $\theta > 0 \Rightarrow$ *There exists* $t_\theta \geq 0$ *with* $\gamma(\theta)(t) = \omega(|t| - M + t_\theta)$ *for all* $|t| \geq M$. t_θ *is a continuous function of* θ.

Proof: The Hamiltonian $\frac{1}{2}\omega'(t)^2 - \frac{1}{2}\omega(t)^2 + V(\omega(t))$ equals zero for all t, so

$$\omega(0)^2 = 2V(\omega(0)). \tag{2.2}$$

By $(W_1) - (W_3)$, $q^2 < 2V(q)$ for all $0 < q < \omega(0)$, so the integrand in $I(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}(\omega')^2 + \frac{1}{2}\omega^2 - V(\omega)\, dt$ is positive for all $t \neq 0$. Define

$$\gamma(\theta)(t) = \begin{cases} 0, & \theta = 0; \\ \omega(\frac{1}{\theta} - 6 + |t|), & 0 < \theta \leq \frac{1}{6}. \end{cases} \tag{2.3}$$

γ is continuous on $[0,1/6]$, $\gamma(1/6)=\omega$, and for all $\theta \in (0,1/6)$,

$$I(\gamma(\theta)) = 2\int_{\frac{1}{\theta}-6}^{\infty} \frac{1}{2}\omega'^2 + \frac{1}{2}\omega^2 - V(\omega) < I(\omega). \tag{2.4}$$

By (V_3) and (2.2),

$$\begin{aligned}\omega(0) - V'(\omega(0)) \le \omega(0) - \frac{\mu V(\omega(0))}{\omega(0)} &= \\ = \frac{\omega(0)^2 - \mu V(\omega(0))}{\omega(0)} &= -\frac{1}{2}(\mu-2)\omega(0) < 0.\end{aligned} \tag{2.5}$$

Let $\alpha > 0$ be small enough so that for all $\omega(0) \le q \le \omega(0)+\alpha$,

$$q - V'(q) \le -\frac{1}{4}(\mu-2)\omega(0). \tag{2.6}$$

$\omega(0)^2 - V(\omega(0)) = 0$, so the Mean Value Theorem implies that for all $\omega(0) \le q \le \omega(0)+\alpha$,

$$\frac{1}{2}q^2 - V(q) \le -\frac{1}{4}(\mu-2)\omega(0)(q-\omega(0)). \tag{2.7}$$

Define

$$M = \max\Big(\sqrt{\frac{8\alpha}{(\mu-2)\omega(0)}}, \frac{24c_0}{(\mu-2)\omega(0)\alpha}\Big). \tag{2.8}$$

For $\theta \in [1/6,1/3]$, let $\gamma(\theta)$ be ω with a horizontal segment of height $\omega(0)$ inserted in the center, with the segment growing from length 0 to M as θ increases from 1/6 to 1/3. That is,

$$\gamma(\theta)(t) = \begin{cases} \omega(0), & |t| \le (6\theta-1)M; \\ \omega(|t|-(6\theta-1)M), & |t| \ge (6\theta-1)M. \end{cases} \tag{2.9}$$

Since $\omega(0)^2 = 2V(\omega(0))$, for each $\theta \in [1/6,1/3]$ we have

$$\begin{aligned} I(\gamma(\theta)) = \int_{-\infty}^{-(6\theta-1)M} & \frac{1}{2}\gamma(\theta)'(t)^2 + \frac{1}{2}\gamma(\theta)(t)^2 - V(\gamma(\theta)(t))\,dt + \\ & + \int_{-(6\theta-1)M}^{(6\theta-1)M} \frac{1}{2}\gamma(0)^2 - V(\gamma(0))\,dt + \\ & + \int_{(6\theta-1)M}^{\infty} \frac{1}{2}\gamma(\theta)'(t)^2 + \frac{1}{2}\gamma(\theta)(t)^2 - V(\gamma(\theta)(t))\,dt = \\ = \int_{-\infty}^{0} & \frac{1}{2}\omega'^2 + \omega^2 - V(\omega) + \int_0^{\infty} \frac{1}{2}\omega'^2 + \omega^2 - V(\omega) = \\ = I(\omega) & = c_0. \end{aligned} \tag{2.10}$$

Next, we will deform $\gamma(1/3)$ so that it is only piecewise linear on $[-M, M]$. For ease in notation, for $s \in [0, \alpha]$ let us temporarily define u_s by

$$u_s(t) = \begin{cases} \omega(|t| - M), & |t| \geq M; \\ \omega(0) + s - s|t|/M, & |t| \leq M. \end{cases} \tag{2.11}$$

Now

$$\begin{aligned} I(u_s) = \int_{-\infty}^{-M} \frac{1}{2}{u_s'}^2 + \frac{1}{2}u_s^2 - V(u_s) + \int_{-M}^{M} \frac{1}{2}{u_s'}^2 + \frac{1}{2}u_s^2 - V(u_s) + \qquad (2.12) \\ \int_{M}^{\infty} \frac{1}{2}{u_s'}^2 + \frac{1}{2}u_s^2 - V(u_s) = \\ = \int_{-\infty}^{0} \frac{1}{2}\omega'^2 + \frac{1}{2}\omega^2 - V(\omega) + \int_{-M}^{M} \frac{1}{2}{u_s'}^2 + \frac{1}{2}u_s^2 - V(u_s) + \\ + \int_{0}^{\infty} \frac{1}{2}\omega'^2 + \frac{1}{2}\omega^2 - V(\omega) = \\ = I(\omega) + \int_0^M {u_s'}^2 + u_s^2 - 2V(u_s) = \\ = c_0 + \int_0^M \left(\frac{s}{M}\right)^2 + (\omega(0) + s - \frac{s}{M}t)^2 - 2V(\omega(0) + s - \frac{s}{M}t)\, dt \\ = c_0 + \frac{s^2}{M} + \frac{2M}{s}\int_{\omega(0)}^{\omega(0)+s} \frac{1}{2}x^2 - V(x)\, dx \leq \\ = c_0 + \frac{s^2}{M} - \frac{(\mu - 2)\omega(0)M}{2s}\int_{\omega(0)}^{\omega(0)+s} x - \omega(0)\, dx = \end{aligned}$$

(by (2.7))

$$= c_0 + \frac{s^2}{M} - \frac{1}{4}(\mu - 2)\omega(0)sM.$$

For $s \in [0, \alpha]$,

$$I(u_s) \leq c_0 + \frac{s}{M}(\alpha - \frac{1}{4}(\mu - 2)\omega(0)M^2) \leq c_0 - \frac{s\alpha}{M} \leq c_0. \tag{2.13}$$

Also,

$$\begin{aligned} I(u_\alpha) &\leq c_0 + \frac{\alpha^2}{M} - \frac{1}{4}(\mu - 2)\alpha\omega(0)M \qquad (2.14) \\ &\leq c_0 + \frac{1}{8}(\mu - 2)\alpha\omega(0)M - \frac{1}{4}(\mu - 2)\alpha\omega(0)M \\ &= c_0 - \frac{1}{8}(\mu - 2)\alpha\omega(0)M \leq c_0 - 3c_0 = -2c_0. \end{aligned}$$

Now we can finish defining γ. For $\theta \in [1/3, 1/2]$, define

$$\gamma(\theta) = u_{6\alpha(\theta - \frac{1}{3})} \tag{2.15}$$

and for $\theta \in [1/2, 1]$, define

$$\gamma(\theta) = u_\alpha. \tag{2.16}$$

The construction of γ is complete.

3. A Substitute for the Nehari Manifold

Let the gradient vector flow η be as described in the Introduction. That is, let ∇I denote the gradient of I; $(\nabla I(u), w) = I'(u)w$ for all $u, w \in W^{1,2}(\mathbf{R}, \mathbf{R}^N)$. Here, $(\cdot, \cdot)$ is the usual inner product defined by $(u, w) = \int_{-\infty}^{\infty} u'(t) \cdot v'(t) + u(t)v(t)\, dt$. $\nabla I(u)$ exists because of the Riesz Representation Theorem, and ∇I is a C^1 vector field. Let $\varphi : W^{1,2}(\mathbf{R}, \mathbf{R}^N) \to [0, 1]$ be locally Lipschitz, with $I(u) \geq -1 \Rightarrow \varphi(u) = 1$ and $I(u) \leq -2 \Rightarrow \varphi(u) = 0$. Let η be the solution of the initial value problem

$$\frac{d\eta}{ds} = -\varphi(\eta)\nabla J(\eta), \;\; \eta(u, 0) = u. \tag{3.1}$$

Lemma 3.2 *η is well-defined on* $\mathbf{R}^+ \times W^{1,2}(\mathbf{R}, \mathbf{R}^N)$

Proof: Suppose that $u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N)$ and $\eta(s, u)$ is not defined for all $s > 0$. Since φ and ∇I are locally Lipschitz, there exist $\bar{s} > 0$ and a sequence $(s^n)_{n\geq 1}$ with $s^n \to \bar{s}$ and $\|\nabla I(\eta(s^n, u)\| \to \infty$. Since I' is bounded on bounded subsets of $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$, $\|\eta(s^n, u)\| \to \infty$. Clearly $I(u) \geq -2$.

Let

$$P = \|u\| + 1 + \sqrt{\frac{4\mu}{\mu - 2}|I(u)|} + \frac{8(I(u) + 2)}{\mu - 2} \tag{3.3}$$

Let $\eta \equiv \eta(s) \equiv \eta(s, u)$. There exist $0 < s_1 < s_2$ with $\|\eta(s_1)\| = P$, $\|\eta(s_2)\| \geq 2P$, and $P < \|\eta(s)\| < 2P$ for all $s_1 < s < s_2$. For all $s_1 < s < s_2$,

$$\begin{aligned} I'(\eta)\eta = \|\eta\|^2 - \int_{-\infty}^{\infty} V'(\eta)\eta \leq \|\eta\|^2 - \mu\int_{-\infty}^{\infty} V(\eta) = \\ = \mu I(\eta) - \frac{1}{2}(\mu - 2)\|\eta\|^2 \leq \mu|I(\eta)| - \frac{1}{2}(\mu - 2)\|\eta\|^2 < \\ < \frac{1}{4}(\mu - 2)P^2 - \frac{1}{2}(\mu - 2)P^2 = -\frac{1}{4}(\mu - 2)P^2, \end{aligned} \tag{3.4}$$

so

$$\|I'(\eta)\| \geq \frac{-I'(\eta)\eta}{\|\eta\|} > \frac{1}{8}(\mu - 2)P. \tag{3.5}$$

Now

$$I(u)+2 \geq I(\eta(s_1)) - I(\eta(s_2)) = -\int_{s_1}^{s_2} \frac{d}{ds} I(\eta(s))\, ds = \tag{3.6}$$
$$= \int_{s_1}^{s_2} \varphi(\eta) \|I'(\eta(s))\|^2\, ds \geq \frac{1}{8}(\mu-2)P \int_{s_1}^{s_2} \varphi(\eta)\|I'(\eta)\|\, ds \geq$$
$$\geq \frac{1}{8}(\mu-2)\int_{s_1}^{s_2} \|\frac{d\eta}{ds}\|\, ds \geq \frac{1}{8}(\mu-2)\|\int_{s_1}^{s_2} \frac{d\eta}{ds}\, ds\| =$$
$$= \frac{1}{8}(\mu-2)\|\eta(s_2)-\eta(s_1)\| \geq \frac{1}{8}(\mu-2)P.$$

Therefore

$$P \leq \frac{8(I(u)+2)}{\mu-2}. \tag{3.7}$$

This contradicts the definition of P. Lemma 3.2 is proven.

♠

Let $\mathcal{B}$ and $\partial\mathcal{B}$ be as defined in (1.14)-(1.15) in the Introduction. Here some properties of $\partial\mathcal{B}$ are proven. First, it is well-known that any Palais-Smale sequence for I is bounded in norm. The following lemma gives a formula that we will neeed for the bound.

LEMMA 3.8 *For all $u \in E$,*

$$\|u\| \leq \frac{2\|I'(u)\| + \sqrt{2\mu(\mu-2)\max(0, I(u))}}{\mu-2}.$$

Proof:

$$-\|I'(u)\|\ \|u\| \leq I'(u)u = \|u\|^2 - \int_{\mathbf{R}} h(t)W'(t,u)u\, dt \leq \tag{3.9}$$
$$\leq \|u\|^2 - \mu\int_{\mathbf{R}} W(t,u)\, dt = \mu I(u) - (\frac{\mu-2}{2})\|u\|^2,$$

so

$$(\frac{\mu-2}{2})\|u\|^2 - \|I'(u)\|\ \|u\| - \mu I(u) \leq 0. \tag{3.10}$$

Applying the quadratic formula to (3.10), and the inequality $\sqrt{A^2+B^2} \leq |A|+|B|$, yields

$$\|u\| \leq \frac{\|I'(u)\| + \sqrt{\|I'(u)\|^2 + 2\mu(\mu-2)\max(0, I(u))}}{\mu-2} \leq \tag{3.11}$$
$$\leq \frac{2\|I'(u)\| + \sqrt{2\mu(\mu-2)\max(0, I(u))}}{\mu-2.}$$

♠

Call a set $A \subset W^{1,2}(\mathbf{R},\mathbf{R}^N)$ *forward-η-invariant* if for all $s > 0$ and $u \in A$, $\eta(s,u) \in A$. Now

LEMMA 3.12

(i) $\mathcal{B}$ *is an open neighborhood of* $0 \in W^{1,2}(\mathbf{R},\mathbf{R}^N)$.

(ii) $\inf_{\partial\mathcal{B}} I > 0$

(iii) $\partial\mathcal{B}$ *is forward-η-invariant.*

(iv) For any $K > 0$, *the set* $\partial\mathcal{B} \cap \{u \in W^{1,2}(\mathbf{R},\mathbf{R}^N \mid I(u) < K\}$ *is bounded.*

Proof: (i): let $r_0 > 0$ be small enough so that for all $t \in \mathbf{R}$ and $v \in \mathbf{R}^N$ with $|v| \le r_0$,

$$W(t,v) \le \frac{1}{6}|v|^2 \text{ and } W'(t,v)v \le \frac{1}{2}|v|^2. \tag{3.13}$$

Let $\|u\| \le r_0$, and $\eta \equiv \eta(s) \equiv \eta(s,u)$. Then $\|u\|_{L^\infty} \le r_0$, and

$$\frac{d}{ds}\|\eta\|^2 = -2I'(\eta)(\eta) = -2\|\eta\|^2 + 2\int_{-\infty}^{\infty} W'(t,\eta)\eta \le -\|\eta\|^2, \tag{3.14}$$

so $\|\eta(s)\|^2 \to 0$ as $s \to \infty$, and $u \in \mathcal{B}$. Thus $\mathcal{B}$ contains the ball $B_{r_0}(0) \equiv \{w \in W^{1,2}(\mathbf{R},\mathbf{R}^N) \mid \|w\| < r_0\}$, an open neighborhood of 0. Now let $u \in \mathcal{B}$. For some $s^* > 0$, $\eta(s^*,u) \in B_{r_0}(0)$. For small enough $r > 0$, $\|w - u\| < r$ implies $\eta(s^*,w) \in B_{r_0}(0)$, and $\eta(s(\eta(s^*,w)) = \eta(s+s^*,w) \to 0$ as $s \to \infty$, so $w \in \mathcal{B}$. So $B_r(u) \equiv \{w \mid \|w - u\| < r\}$ is an open neighborhood of u that is contained in $\mathcal{B}$.

(ii): $\partial\mathcal{B}$ is nonempty, for if $I(u) < 0$, then $u \notin \mathcal{B}$. Let $u \in \partial\mathcal{B}$. There exists a sequence $(u_n) \subset \mathcal{B}$ with $u_n \to u$. Let r_0 be as in the proof of (i). $\|u_n\| \ge r_0$ for large n. Since $\eta(s,u_n) \to 0$, there exists s_n with $\|\eta(s_n,u_n)\| = r_0$. Then $|\eta(s_n,u_n)(t)| \le r_0$ for all $t \in \mathbf{R}$. By the definition of r_0,

$$\begin{aligned} I(\eta(s_n,u_n)) = \int_{-\infty}^{\infty} & \frac{1}{2}\eta(s_n,u_n)'(t)^2 + \frac{1}{2}\eta(s_n,u_n)(t)^2 \\ & - W(t,\eta(s_n,u_n)(t))\,dt \ge \frac{1}{3}\|\eta(s_n,u_n)\|^2 = r_0^2/3, \end{aligned} \tag{3.15}$$

so $I(u_n) \ge r_0^2/3$, $I(u) \ge r_0^2/3$, and (ii) is proven.

(iii): Let $u \in \mathcal{B}$ and $s_1 > 0$. Since $\eta(s,u) \to 0$ as $m \to \infty$, $\eta(s+s_1,u) = \eta(s,\eta(s_1,u)) \to 0$ as $s \to \infty$, and $\eta(s_1,u) \in \mathcal{B}$. Thus $\mathcal{B}$ is forward-η-invariant. Next, let $u \in \partial\mathcal{B}$ and $s > 0$. Since $\mathcal{B}$ is open, $u \notin \mathcal{B}$. $\eta(s,u)$ is not in $\mathcal{B}$, for if it were, the definition of $\mathcal{B}$ would imply $u \in \mathcal{B}$. u is in the closure of $\mathcal{B}$, so let $(u_m) \subset \mathcal{B}$ with $u_m \to u$. $\eta(s,u_m) \to \eta(s,u)$ and $\eta(s,u_m) \in \mathcal{B}$, so $\eta(s,u)$ belongs to the closure of $\mathcal{B}$.

(iv) If suffices to show that for any $K > 0$, the set $\mathcal{B} \cap \{u \in W^{1,2}(\mathbf{R}, \mathbf{R}^N \mid I(u) < K\}$ is bounded. We use an "annulus" argument. Let $K > 0$, and let

$$P = 1 + \frac{2\mu K}{\mu - 2} + \frac{16K^2}{(\mu-2)^2}. \tag{3.16}$$

Let $u \in \partial\mathcal{B}$ with $I(u) \le K$. Assume $\|u\| > 2P$. This will lead to a contradiction.

By the definition of $\mathcal{B}$ and the fact that $\mathcal{B}$ is open, it is clear that $I(u) \ge 0$. For any $w \in E$ with $I(w) \le 0$ and $\|w\| \ge P$, Lemma 3.8 gives

$$\begin{aligned} \|I'(w)\| &\ge \frac{1}{2}\big((\mu-2)\|w\| - \sqrt{2\mu(\mu-2)I(w)}\ \big) \ge \\ &\ge \frac{1}{2}\big((\mu-2)P - 2\mu\sqrt{K}\big) \ge \frac{\mu-2}{4}P. \end{aligned} \tag{3.17}$$

By Lemma 3.2, $\eta(s,u)$ is well-defined for all $s > 0$. Since $I(\eta(s,u)) > 0$ for all $s > 0$, and $\frac{d}{ds}I(\eta(s,u)) = -\|I'(\eta(s,u))\|^2$, (3.17) implies that $\|\eta(s^*,u)\| \le P$ for some $s^* > 0$. Let $\eta \equiv \eta(s) \equiv \eta(s,u)$. There exist $0 < s_1 < s_2$ with $\|\eta(s_1)\| = 2P$, $\|\eta(s_2)\| = P$, and $\|\eta(s_1)\| \in (P, 2P)$ for all $s \in (P, 2P)$. Then by (3.17),

$$\begin{aligned} K \ge I(\eta(s_1)) - I(\eta(s_2)) &= -\int_{s_1}^{s_2} \frac{d}{ds} I(\eta(s))\, ds = \\ &= \int_{s_1}^{s_2} \|I'(\eta(s))\|^2\, ds \ge (s_2 - s_1)\frac{(\mu-2)^2}{16}P^2. \end{aligned} \tag{3.18}$$

But

$$\begin{aligned} P \le \|\eta(s_1) - \eta(s_2)\| &= \|\int_{s_1}^{s_2} \frac{d\eta}{ds}\, ds\| \le \int_{s_1}^{s_2} \|\frac{d\eta}{ds}\|\, ds = \\ &= \int_{s_1}^{s_2} \|I'(\eta)\|\, ds \le \sqrt{s_2 - s_1} \cdot \sqrt{\int_{s_1}^{s_2} \|I'(\eta)\|^2\, ds} = \end{aligned} \tag{3.19}$$

(by the Cauchy-Schwarz Inequality)

$$\begin{aligned} &= \sqrt{s_2 - s_1} \cdot \sqrt{\int_{s_1}^{s_2} \frac{d}{ds} I(\eta(s))\, ds} = \\ &= \sqrt{s_2 - s_1} \cdot \sqrt{I(\eta(s_1)) - I(\eta(s_2))} \le \sqrt{(s_2 - s_1)K}. \end{aligned}$$

(3.18)-(3.19) give

$$\frac{P^2}{K} \le s_2 - s_1 \le \frac{16K}{(\mu-2)^2 P^2}, \qquad P^4 \le \frac{16K^2}{(\mu-2)^2}. \tag{3.20}$$

This contradicts the definition of P. Lemma 3.12 is proven.

♠

Note: it is unclear whether $\partial\mathcal{B}$ must be homeomorphic to the unit ball of $W^{1,2}(\mathbf{R},\mathbf{R}^N)$.

Define $I_0 \in C^2(W^{1,2}(\mathbf{R},\mathbf{R}^N),\mathbf{R})$ by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \int_{-\infty}^{\infty} V(|u|)\,dt. \tag{3.21}$$

Roughly, this "autonomous" functional satisfies $I_0(u) \approx I(u)$ if the bulk of u is supported far from 0. The reason that we can consider the scalar equation (1.6) to be the problem at infinity for (1.1) is that any nonzero critical point of I_0 has the form $\tau_a\omega\,\mathbf{u}$ for some $a \in \mathbf{R}$ and unit vector $\mathbf{u} \in \mathbf{R}^N$. To prove this, it suffices to show that all critical points u of I_0 are radial, that is, $u(t) = g(t)\mathbf{u}$ for some scalar function g and unit vector $u \in \mathbf{R}^N$. Let u be a nontrivial critical point of I_0, satisfying

$$-u'' + u = \nabla(V(|u|)) = \frac{V'(|u|)u}{|u|}. \tag{3.22}$$

Consider the quantity $(u\cdot u')^2 - |u|^2|u'|^2$. This expression tends to zero as $t \to \pm\infty$. If it equals zero for some t, then $u'(t)$ and $u(t)$ are parallel (this is the equality case of the Cauchy-Schwarz Inequality). So it suffices to show that $\frac{d}{dt}[(u\cdot u')^2 - |u|^2|u'|^2]$ is always zero.

$$\begin{aligned}
&\frac{d}{dt}\big[(u\cdot u')^2 - |u|^2|u'|^2\big] = \qquad (3.23)\\
&\quad = 2(u\cdot u')(|u'|^2 + u\cdot u'') - 2(u\cdot u')|u'|^2 - 2|u|^2(u'\cdot u'') =\\
&\quad = 2\big[(u\cdot u')(u\cdot u'') - |u|^2(u'\cdot u'')\big] =\\
&\quad = 2\big[(u\cdot u')(u\cdot(u - V'(|u|)u/|u|)) - |u|^2(u'\cdot(u - V'(|u|)u/|u|))\big] =\\
&\quad = 2\big[(u\cdot u')(|u|^2 - |u|V'(|u|)) - |u|^2(u'\cdot u - \frac{V'(|u|)u\cdot u'}{|u|})\big] = 0.
\end{aligned}$$

It is well-known that the functional I does not satisfy the Palais-Smale condition, that is, a Palais-Smale sequence need not have a convergent subsequence. A Palais-Smale sequence is a sequence $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ with $I'(u_n) \to 0$ and $(I(u_n))$ convergent. The proposition below states that a Palais-Smale sequence "splits" into the sum of a critical point of I and translates of critical points of I_0:

Proposition 3.24 *If $(u_n) \subset W^{1,2}(\mathbf{R},\mathbf{R}^N)$ with $I'(u_n) \to 0$ and $I(u_n) \to a > 0$, then there exist $k \geq 0$, $v_0, v_1, \ldots, v_k \in W^{1,2}(\mathbf{R},\mathbf{R}^N)$, and sequences $(t_m^i)_{m\geq 1}^{1\leq i\leq k} \subset \mathbf{R}$, such that*

(i) $I'(v_0) = 0$

(ii) $I_0'(v_i) = 0$ *for all* $i = 1, \ldots, k$

and along a subsequence (also denoted (u_n))

(iii) $\|u_n - (v_0 + \sum_{i=1}^{k} \tau_{t_n^i} v_i)\| \to 0$ *as* $n \to \infty$

(iv) $|t_n^i| \to \infty$ *as* $n \to \infty$ *for all* $i = 1, \ldots, k$

(v) $t_n^{i+1} - t_n^i \to \infty$ *as* $n \to \infty$ *for all* $i = 1, \ldots, k-1$

(vi) $I(v_0) + kc_0 = a$

A proof for the case of t-periodic W is found in [CR], and essentially the same proof works here. Similar propositions for nonperiodic coefficient functions, for both ODE and PDE, are found in [CMN], [AM], and [S2], for example. All are inspired by the "concentration-compactness" theorems of P. -L. Lions ([L]).

From now on, assume

$$I \text{ has no critical values in the interval } (0, c_0]. \tag{3.25}$$

Define the (continuous) "location" functional $\mathcal{L} : L^2(\mathbf{R},\mathbf{R}^N) \setminus \{0\} \to \mathbf{R}$ by

$$\int_{-\infty}^{\infty} |u|^2 \tan^{-1}(t - \mathcal{L}(u))\, dt = 0 \tag{3.26}$$

Roughly, $\mathcal{L}$ tells where along the real line a nonzero function is concentrated. If u is even, then $\mathcal{L}(u) = 0$. Define

(3.28) $$\tilde{c} = \inf\{I(u) \mid u \in \partial\mathcal{B},\ \mathcal{L}(u) = 0\}.$$

Under the assumption (3.25), we claim:

$$\tilde{c} > c_0. \tag{3.28}$$

Proof: If $\tilde{c} < c_0$, then there exists $u \in \partial\mathcal{B}$ with $I(u) < c_0$. By arguments of [CMN], the sequence $(\eta(n,u))$ is a Palais-Smale sequence. By Proposition 3.24,

$(\eta(n,u))$ converges along a subsequence to a critical point v of I with $0 < I(u) < c_0$, contradicting (3.25). Next, suppose $\tilde{c} = c_0$. Then there exists $(u_n) \subset \partial\mathcal{B}$ with $\mathcal{L}(u_n) = 0$ and $I(u_n) \to \tilde{c}$. (u_n) is bounded, by Lemma 3.12(iv). If, along a subsequence, $\|I'(u_n)\| > p > 0$, then by arguments of [CMN], since I' is Lipschitz on bounded subsets of $W^{1,2}(\mathbf{R}, \mathbf{R}^N)$, $I(\eta(1, u_n)) < \tilde{c} = c_0$ for large enough n. Then, like above, for large n, $(\eta(m, u_n))_{m\geq 1}$ is a Palais-Smale sequence converging to a critical point v of I with $0 < I(v) < c_0$, contrary to (3.25). Thus $\|I'(u_n)\| \to 0$. Since $\mathcal{L}(u_n) = 0$ for all n, Proposition 3.24 implies that (u_n) converges strongly to a critical point v of I with $I(v) = c_0$. This contradicts assumption (3.25). Claim (3.28) is proven.

4. The Minimax Argument: Interacting Tails

We are almost ready to define $\mathcal{G}$, from (1.16). First we need to define R_1 precisely. Let $r_1 > 0$ be small enough so

$$0 \leq q \leq r_1 \Rightarrow V(q) \leq \frac{1}{18} q^2. \tag{4.1}$$

Let $R_0 > M$ where M is from Proposition 2.1, and big enough so that

$$\gamma(\theta)(t) < r_1 \tag{4.2}$$

for all $\theta \in [0,1]$ and $|t| \geq R_0$. Let $R_1 > R_0$ and let R_1 be large enough so that for all $u \in \mathbf{R}^N$,

$$|t| \geq R_1 \Rightarrow W(t,u) \leq 3V(u). \tag{4.3}$$

This is possible by (W_4). By Proposition 2.1(iii),

$$\inf\{\gamma(\theta)(t) \mid J(\gamma(\theta)) \geq c_0/2,\ |t| \leq R_0\} > 0. \tag{4.4}$$

Let R_1 be big enough so that

$$\sup\{\gamma(\theta)(t) \mid \theta \in [0,1],\ |t| \geq R_1\} < \inf\{\gamma(\theta)(t) \mid J(\gamma(\theta)) \geq c_0/2,\ |t| \leq R_0\}. \tag{4.5}$$

By (W_6), $2/\delta < 1 - 2/\mu$, so we may choose $\epsilon > 0$ and d with

$$\epsilon < \frac{1}{2}, \qquad \frac{2}{\delta} < d < 1 - \frac{2}{\mu(1-\epsilon)}. \tag{4.6}$$

Let C be large enough so that for all $\theta \in [0,1]$ and $t \in \mathbf{R}$,

$$\gamma(\theta)(t) \le Ce^{-(1-\epsilon)|t|}. \tag{4.7}$$

This is possible by Proposition 2.1(v) and (vi), and since ω satisfies $-\omega''+\omega = V'(\omega)$ with $V'(q) = o(q)$ as $q \to 0$.

Let B be large enough so that for all $\theta \in [0,1]$ and $t \in \mathbf{R}$,

$$V(\gamma(\theta)(t)) \le BV(\gamma(\theta)(t))^{\mu}. \tag{4.8}$$

This is possible by (V_3).

Let $l > 0$ be small enough so that for all $\theta \in [0,1]$,

$$J(\gamma(\theta)) \ge c_0/2 \Rightarrow \gamma(\theta)(t) > le^{-t} \text{ for all } |t| > R_0. \tag{4.9}$$

This is possible by Proposition 2.1(vii), since ω satisfies $\omega'' = \omega - V(\omega)$.

Now $\delta d > 2$ and $\mu(1-d)(1-\epsilon) > 2$, so we may choose R_1 to be large enough so

$$\frac{1}{3}l^2 e^{-2R_1} > 2ABC^{\mu}(e^{-\delta d R_1} + e^{-\mu(1-d)(1-\epsilon)R_1}). \tag{4.10}$$

Finally, assume that R_1 is large enough that for all $\theta_1, \theta_2 \in [0,1]$,

$$I(\max(\tau_{-R_1}\gamma(\theta_1), \tau_{R_1}\gamma(\theta_2))\mathbf{e}_1) < J(\gamma(\theta_1)) + J(\gamma(\theta_2)) + \min(\frac{c_0}{4}, \frac{\tilde{c} - c_0}{2}). \tag{4.11}$$

Now let $\mathcal{G}$, m, and G_0 be defined as in (1.6)-(1.8). We will prove:

PROPOSITION 4.12

(i) $m > c_0$

(ii) $m > \sup_{G \in \mathcal{G}} \max_{(x,y) \in \partial[0,1]^2} I(G(x,y))$

(iii) $m < 2c_0$

Then, by standard deformation arguments as in [R], there exists a Palais-Smale sequence $(u_n) \subset W^{1,2}(\mathbf{R}, \mathbf{R}^N)$ with $I'(u_n) \to 0$ and $I(u_n) \to m$. By Proposition 3.24, (u_n) has a subsequence converging to v, a critical point of I with $I(v) = m$, and Theorem 1.2 follows.

To prove Proposition 4.12(i), let $G \in \mathcal{G}$ and suppose that g is an arbitrary path from the bottom to the top of $[0,1]^2$. That is, $g \in C([0,1],[0,1]^2)$ with $\pi_2(g(0)) = 0$

and $\pi_2(g(1)) = 1$, where π_2 denotes projection onto the second coordinate. Define $\gamma_g \in \Gamma$ by $\gamma_g(\theta) = G(g(\theta))$. Since $\gamma_g \in \Gamma$, $\gamma_g(\theta) \in \partial\mathcal{B}$ for some $\theta \in [0,1]$.

Since g is an arbitrary path from the bottom to the top of $[0,1]^2$, there exists a connected set $D \subset [0,1]^2$ with $(x,y) \in D \Rightarrow G(x,y) \in \partial\mathcal{B}$, connecting the left and right sides of $[0,1]^2$. That is, D is a connected subset of $[0,1]^2$ with $(0,y_0), (1,y_1) \in D$ for some $y_0, y_1 \in [0,1]$, and $(x,y) \in D \Rightarrow G(x,y) \in \partial\mathcal{B}$. $G(0,y_0) = \tau_{-R_1}\gamma(y_0)\,\mathbf{e}_1$ with $y_0 > 0$, and $\gamma(y_0)$ is a nonzero even function of t, so $\mathcal{L}(G(0,y_0)) = -R_1$. Likewise, $\mathcal{L}(G(1,y_1)) = R_1$. Since $\mathcal{L}$ is continuous and D is connected, $\mathcal{L}(G(x^*,y^*)) = 0$ for some $(x^*,y^*) \in D$. $I(G(x^*,y^*)) \geq \tilde{c}$ by the definition of $\tilde{c}$. Since G is an arbitrary element of $\mathcal{G}$, $m \geq \tilde{c} > c_0$.

To prove Proposition 4.12(ii), note that for any $G \in \mathcal{G}$ and $y \in [0,1]$,

$$\begin{aligned} I(G(0,y)) &= I(\tau_{-R_1}\gamma(y)\mathbf{e}_1) = I(\max(\tau_{-R_1}\gamma(y),0)\mathbf{e}_1) \leq \\ &\leq J(\gamma(y)) + 0 + (\tilde{c} - c_0)/2 \\ &\leq c_0 + 0 + (\tilde{c} - c_0)/2 = \frac{c_0 + \tilde{c}}{2} < \tilde{c}. \end{aligned} \tag{4.13}$$

Similarly, $I(G(1,y)) \leq (c_0 + \tilde{c})/2$. For all $x \in [0,1]$, $I(G(x,0)) = I(0) = 0$, and since either x or $1-x$ is $\geq 1/2$, (4.11) gives

$$\begin{aligned} I(G(x,1)) &= I(\max(\tau_{-R_1}\gamma(1-x), \tau_{R_1}\gamma(x))\mathbf{e}_1) \leq \\ &\leq J(\gamma(1-x)) + J(\gamma(x)) + \frac{c_0}{4} \leq c_0 - 2c_0 + \frac{c_0}{4} < 0. \end{aligned} \tag{4.14}$$

Finally, we must prove Proposition 4.12(iii). Let $G_0 \in \mathcal{G}$ be defined as in (1.18). First, suppose that

$$J(\gamma((1-x)y)) \leq c_0/2 \text{ or } J(\gamma(xy)) \leq c_0/2. \tag{4.15}$$

Then by (4.11),

$$I(G_0(x,y)) \leq J(\gamma((1-x)y)) + J(\gamma(xy)) + c_0/4 \leq c_0 + c_0/2 + c_0/4 < 2c_0. \tag{4.16}$$

So suppose from now on that

$$J(\gamma((1-x)y)) > c_0/2 \text{ and } J(\gamma(xy)) > c_0/2. \tag{4.17}$$

For ease of notation, let $u_1 = \gamma((1-x)y)$ and $u_2 = \gamma(xy)$. We must show $2c_0 - I(G_0(x,y))$ is positive:

$$
\begin{aligned}
2c_0 - I(G_0(x,y)) &= 2c_0 - I(\max(\tau_{-R_1}u_1, \tau_{R_1}u_2)\mathbf{e}_1) \geq \qquad (4.18)\\
&\geq J(\tau_{-R_1}u_1) + J(\tau_{R_1}u_2) - I(\max(\tau_{-R_1}u_1, \tau_{R_1}u_2)\mathbf{e}_1) = \\
&\big(J(\tau_{-R_1}u_1) - I(\tau_{-R_1}u_1\mathbf{e}_1)\big) + \\
&\big(J(\tau_{R_1}u_2) - I(\tau_{R_1}u_2\mathbf{e}_1)\big) + \\
&\big(I(\tau_{-R_1}u_1\mathbf{e}_1) + I(\tau_{R_1}u_2\mathbf{e}_1) - I(\max(\tau_{-R_1}u_1, \tau_{R_1}u_2)\mathbf{e}_1)\big) \equiv \\
&X_1 + X_2 + Y.
\end{aligned}
$$

We will show $X_1+X_2+Y > 0$. By (W_6) with $W(t,u)-V(u) \geq -AV(|u|)e^{-\delta|t|}$,(4.8), and (4.7),

$$
\begin{aligned}
X_2 &= \int_{-\infty}^{\infty} W(t, \tau_{R_1}u_2\mathbf{e}_1) - V(\tau_{R_1}u_2)\,dt \geq \qquad (4.19)\\
&\geq -A\int_{-\infty}^{\infty} V(\tau_{R_1}u_2)e^{-\delta|t|}\,dt \geq -AB\int_{-\infty}^{\infty} (\tau_{R_1}u_2)^{\mu}e^{-\delta|t|}\,dt = \\
&= -AB\int_{-\infty}^{\infty} u_2(t-R_1)^{\mu}e^{-\delta|t|}\,dt \geq -ABC^{\mu}\int_{-\infty}^{\infty} e^{-\mu(1-\epsilon)|t-R_1|}\,dte^{-\delta|t|}\,dt.
\end{aligned}
$$

A similar integral is obtained if instead $W(t,u) - V(u) \geq -A|u|^{\mu}e^{-\delta|t|}$ in (W_6). Estimating the last integral,

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\mu(1-\epsilon)|t-R_1|}e^{-\delta|t|}\,dt &\leq \int_{-\infty}^{dR_1} e^{-\mu(1-\epsilon)|t-R_1|}\,dt + \int_{dR_1}^{\infty} e^{-\delta|t|}\,dt = \qquad (4.20)\\
&= \frac{1}{\mu(1-\epsilon)}e^{-\mu(1-\epsilon)(1-d)R_1} + \frac{1}{\delta}e^{-\delta dR_1} < \\
&< e^{-\mu(1-\epsilon)(1-d)R_1} + e^{-\delta dR_1},
\end{aligned}
$$

so

$$
X_2 \geq -ABC^{\mu}(e^{-\mu(1-\epsilon)(1-d)R_1} + e^{-\delta dR_1}). \qquad (4.21)
$$

Similarly,

$$
X_1 \geq -ABC^{\mu}(e^{-\mu(1-\epsilon)(1-d)R_1} + e^{-\delta dR_1}). \qquad (4.22)
$$

To estimate Y, we must work with the maximum of the functions $\tau_{-R_1}u_1$ and $\tau_{R_1}u_2$. First we establish the following claim.

$$
\begin{gathered}
\text{There exists } t^* \in (-(R_1-R_0), R_1-R_0) \text{ such that} \qquad (4.23)\\
\tau_{-R_1}u_1 \geq \tau_{R_1}u_2 \text{ on } (-\infty, t^*),\\
\tau_{-R_1}u_1(t^*) = \tau_{R_1}u_2(t^*), \text{ and}\\
\tau_{R_1}u_2 \geq \tau_{-R_1}u_1 \text{ on } (t^*, \infty).
\end{gathered}
$$

Proof of claim: Recall that $u_1 = \gamma(xy)$, $u_2 = \gamma((1-x)y)$ where $I(\gamma(xy)) \geq c_0/2$, $I(\gamma((1-x)y)) \geq c_0/2$. To prove Claim (4.23), it suffices to prove

$$\begin{aligned} \tau_{-R_1}u_1 > \tau_{R_1}u_2 &\text{ on } (\infty, -(R_1-R_0]), \\ \tau_{R_1}u_2 > \tau_{-R_1}u_1 &\text{ on } [R_1-R_0,\infty), \\ \tau_{-R_1}u_1 &\text{ is nonincreasing on } [-(R_1-R_0),R_1-R_0], \text{ and} \\ \tau_{R_1}u_2 &\text{ is nondecreasing on } [-(R_1-R_0),R_1-R_0]. \end{aligned} \tag{4.24}$$

u_1 is nonincreasing on $[0,\infty)$ by Proposition 2.1(vi), so $\tau_{-R_1}u_1$ is nonincreasing on $[-R_1,\infty)$, an interval that includes $[-(R_1-R_0),R_1-R_0]$. Likewise, $\tau_{R_1}u_2$ is nondecreasing on $[-(R_1-R_0),R_1-R_0]$.

Let $t \in [R_1-R_0,R_1+R_0]$. Then $t-R_1 \in [-R_0,R_0]$ and $t+R_1 > 2R_1-R_0 > R_1$, so by (4.5),

$$\tau_{-R_1}u_1(t) = u_1(t+R_1) < u_2(t-R_1) = \tau_{R_1}u_2(t). \tag{4.25}$$

So $\tau_{R_1}u_2 > \tau_{-R_1}u_1$ on $[R_1-R_0,R_1+R_0]$.

On $[R_1+R_0,\infty)$, $\tau_{-R_1}u_1$ and $\tau_{R_1}u_2$ both equal right-hand "tails" of ω, with $\tau_{-R_1}u_1(R_1+R_0) < \tau_{R_1}u_2(R_1+R_0)$. Since ω is decreasing on the positive reals, $\tau_{R_1}u_2 > \tau_{-R_1}u_1$ on $[R_1+R_0,\infty)$. More precisely, there exist $t_1,t_2 \geq 0$ such that for all $t \geq R_1+R_0$, $\tau_{R_1}u_2(t) = \omega(t-(R_1+R_0)+t_2)$ and $\tau_{-R_1}u_1(t) = \omega(t-(R_1+R_0)+t_1)$. Since $\tau_{R_1}u_2(R_1+R_0) > \tau_{-R_1}u_1(R_1+R_0)$, $t_2 > t_1$, and for all $t \geq R_1+R_0$,

$$\tau_{R_1}u_2(t) = \omega(t-(R_1+R_0)+t_2) > \omega(t-(R_1+R_0)+t_1) = \tau_{-R_1}u_1(t). \tag{4.26}$$

So $\tau_{R_1}u_2 > \tau_{-R_1}u_1$ on $[R_1+R_0,\infty)$. Similarly, $\tau_{R_1}u_2 > \tau_{-R_1}u_1$ on $(-\infty,-(R_1-R_0)]$. (4.23) and Claim (4.22) are proven.

Now I is defined by $I(u) = \int_{-\infty}^{\infty} \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|u(t)|^2 - W(t,u(t))\,dt$. $\max(\tau_{-R_1}u_1,\tau_{R_1}u_2)$ agrees with $\tau_{-R_1}u_1$ on $(-\infty,t^*)$ and with $\tau_{R_1}u_2$ on (t^*,∞). Therefore,

$$\begin{aligned} Y = I(\tau_{-R_1}u_1\mathbf{e}_1) + I(\tau_{R_1}u_2\mathbf{e}_1) - I(\max(\tau_{-R_1}u_1,\tau_{R_1}u_2)\mathbf{e}_1) = \\ = \int_{-\infty}^{t^*} \frac{1}{2}\tau_{R_1}u_2'(t)^2 + \frac{1}{2}\tau_{R_1}u_2(t)^2 - W(t,\tau_{R_1}u_2(t))\,dt + \\ + \int_{t^*}^{\infty} \frac{1}{2}\tau_{-R_1}u_1'(t)^2 + \frac{1}{2}\tau_{-R_1}u_1(t)^2 - W(t,\tau_{-R_1}u_1(t))\,dt. \end{aligned} \tag{4.27}$$

Both integrals are nonnegative: $|t^*| < R_1 - R_0$, so for all $t \geq t^*$, $t + R_1 > R_0$, and by (4.1)-(4.3), $u_1(t+R_1) < r_1$, $W(t, u_1(t+R_1)) \leq 3V(u_1(t+R_1)) \leq u_1(t+R_1)^2/6$, and the integrand of the second integral is nonnegative. Similarly, the first integral is nonnegative.

If $t^* < 0$, then $R_0 \leq t^* + R_1 \leq R_1$, and we can estimate the second integral in (4.27):

$$\begin{aligned}
&\int_{t^*}^{\infty} \frac{1}{2}\tau_{-R_1} u_1'(t)^2 + \frac{1}{2}\tau_{-R_1} u_1(t)^2 - W(t, \tau_{-R_1} u_1(t))\, dt \geq \qquad (4.28)\\
&\quad \geq \frac{1}{3} \int_{t^*}^{\infty} u_1'(t+R_1)^2 + u_1(t+R_1)^2\, dt = \frac{1}{3} \int_{t^*+R_1}^{\infty} u_1'(t)^2 + u_1(t)^2\, dt \geq \\
&\quad \geq \frac{1}{3} \|u_1\|^2_{L^\infty(t^*+R_1,\infty)} \geq \frac{1}{3} u_1(t^*+R_1)^2 \geq \frac{1}{3} l^2 e^{-2|t^*+R_1|} \geq \frac{1}{3} l^2 e^{-2R_1}
\end{aligned}$$

by (4.9). If $t^* > 0$, then we can estimate the first integral in (4.27), similarly:

$$\int_{-\infty}^{t^*} \frac{1}{2}\tau_{R_1} u_2'(t)^2 + \frac{1}{2}\tau_{R_1} u_2(t)^2 - W(t, \tau_{R_1} u_2(t))\, dt \geq \frac{1}{3} l^2 e^{-2R_1}. \qquad (4.29)$$

Thus

$$Y \geq \frac{1}{3} l^2 e^{-2R_1}. \qquad (4.30)$$

Putting together (4.18), (4.21)-(4.22), and (4.30), it follows that $2c_0 > I(G_0(x, y))$, Proposition 4.12(iii) is proven, and Theorem 1.2 follows.

Open Questions

Many open questions remain. For example, is the constant $2\mu/(\mu - 2)$ in (W_6) optimal? Proving so would require a counterexample. Can the Ambrosetti-Rabinowitz condition (W_3) be weakened? Does a PDE version of Theorem 1.2 hold, with or without the monotonicity condition (1.10)?

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