A Perturbation of a Periodic Hamiltonian System

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1. Introduction

In this paper we examine a Hamiltonian system with a "superquadratic" term, that is, of the form $-u'' + u = f(t)V(u)$, where $V(u)$ behaves like a power greater than 2 of u. We look for solutions homoclinic to zero, or just "homoclinic," that is, solutions u with $|u'(t)| + |u(t)| \to 0$ as $|t| \to \infty$. Such solutions can be difficult to find, due to the lack of compactness of the domain and the subsequent failure of the Palais-Smale condition in variational formulations of the problem (see [1] for an example of such a problem with no homoclinic solutions). The existence of such solutions depends delicately on f and V. It is known that they exist when f is periodic. We will examine a system in which f is a perturbation of a periodic function. Assuming reasonably weak conditions on the periodic problem, we show that the perturbed problem has a solution if the perturbation is small enough. The conditions required of the periodic problem can be verified for some specific examples. We employ variational, mountain-pass techniques. The result almost extends to an analogous elliptic partial differential equation, but subtle differences in the topology of $\mathbf{R}$ and $\mathbf{R}^n$ $(n \geq 2)$ prevent a complete extension. Nevertheless, we find that for a large class of nearly-autonomous elliptic PDE, there exists a nonzero solution decaying at infinity.

Consider the following system containing a periodic term:

$$-u'' + u = h(t)V'(u), \tag{1.0}$$

where $n \geq 1$ and V and h satisfy

(V_1) $V \in C^2(\mathbf{R}^n, \mathbf{R})$

(V_2) $V(0) = 0$, $V'(0) = 0$, $V''(0) = 0$

(V_3) there exists $p > 1$ such that $V''(q)q \cdot q \geq pV'(q) \cdot q > 0$ for all $q \in \mathbf{R}^n \setminus \{0\}$

(V_4) the mapping $q \mapsto V''(q)q$ is locally Lipschitz on $\mathbf{R}^n$

(h_1) $h \in C^1(\mathbf{R}, \mathbf{R})$

(h_2) $h(t) > 0$ for all $t \in \mathbf{R}$

(h_3) h is periodic.

$(V_1) - (V_3)$ imply that $V'(q) \cdot q \geq (p+1)V(q) > 0$ for $q \neq 0$. Therefore $V(q)/|q|^2 \to 0$ as $|q| \to 0$, and $V(q)/|q|^2 \to \infty$ as $|q| \to \infty$. We know of no precedent for (V_4), but (V_4) does not seem overly restrictive, since $(V_1) - (V_4)$ are satisfied by the canonical example $V(q) = |q|^{p+1}$ with $p > 1$.

In [2] it was proven, under slightly more general and weaker assumptions, that (1.0) must have homoclinic solutions. In [3] it was proven that if the solutions of (1.0) obey a certain nondegeneracy condition, then

(1.0) has infinitely many solutions that can be described as "multibumps," that is, close to the sum of several homoclinic solutions which are mainly supported on intervals which are far away from each other. In [4] (again, using weaker and more general hypotheses) it is shown that if h is perturbed a little, then, if a similar nondegeneracy condition on the homoclinic solutions of (1.0) is assumed, the system still has multibump solutions.

The nondegeneracy conditions mentioned above are extremely difficult to verify. This raises an obvious question. Can such conditions be omitted or weakened, or replaced with more easily verifiable conditions? In this paper we achieve the latter for a certain class of Hamiltonian systems. Consider a Hamiltonian system of the form

$$-u'' + u = (h(t) + \epsilon g(t))V'(u), \tag{1.1}$$

where g satisfies

(g_1) $g \in C^1(\mathbf{R}, \mathbf{R})$,

(g_2) $\sup_{t \in \mathbf{R}} |g(t)| < \infty$, and

(g_3) $g(t) \to 0$ as $|t| \to \infty$.

We will show that, if we assume certain conditions on the homoclinic solutions of the unperturbed problem (1.0), then for small enough $|\epsilon|$, (1.1) has a nontrivial homoclinic solution. In order to state these conditions, we need to examine the variational framework of the problem.

Variational Framework and Theorem Statement

Let $E = W^{1,2}(\mathbf{R}, \mathbf{R}^n)$ along with the inner product

$$(u, w) = \int_{\mathbf{R}} \Big(u' \cdot w' + u \cdot w \Big)\, dt \tag{1.2}$$

for $u, w \in E$ and the associated norm $\|u\| \equiv \|u\|_{W^{1,2}(\mathbf{R})}$. Then the functional $I \in C^2(E, \mathbf{R})$ corresponding to (1.0) is

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} h(t)V(u)\, dt. \tag{1.3}$$

The functional $I_\epsilon \in C^2(E, \mathbf{R})$ corresponding to (1.1) is

$$I_\epsilon(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} (h(t) + \epsilon g(t))V(u)\, dt. \tag{1.4}$$

Any critical point v of I (resp. I_ϵ) is a homoclinic solution of (1.0) (resp. (1.1)). We seek nonzero critical points of I_ϵ. Define the set

$$\mathcal{S} = \{u \mid I'(u)u = 0,\ u \neq 0\} \tag{1.5}$$

and

$$c = \inf_{\mathcal{S}} I. \tag{1.6}$$

(Note: it is easy to verify that c is the "mountain-pass" value associated with I). In Section 2 we will see that c must be a critical value of I. The nondegeneracy assumption of [4] (also in [5]) is the following: there exists $\alpha > 0$ such that the set of critical points

$$\{v \mid I'(v) = 0,\ I(v) < c + \alpha\} \text{ is countable.} \tag{1.7}$$

This is a variational analogue of the classical transversality condition, and either one is very difficult to verify or contradict in general. (1.7) fails, for example, if h is a constant, for then (1.0) has uncountable continua of solutions. The only example we know of in which this kind of condition has been verified is in the new paper [6].

Define $\mathcal{K} = \{v \in E \mid I'(v) = 0\}$ and $\mathcal{K}(c) = \{v \in \mathcal{K} \mid I(v) = c\}$. We can now state the theorem:

Theorem 1.8 *If V and h satisfy $(V_1)-(V_3)$ and $(h_1)-(h_3)$, and g satisfies $(g_1)-(g_3)$, and one of the following cases holds:*

Case I: $\mathcal{K}(c)$ has a connected component that is compact

Case II: c is an isolated critical value of I,

then there exists $\epsilon_0 > 0$ with the property that if $|\epsilon| \le \epsilon_0$, then (1.1) has a nontrivial homoclinic solution.

Case I is very similar to a condition found in [7]. Recent work by Alessio and Montecchiari ([8]) suggests that Case I can be verified if h "oscillates slowly," that is, if $\tilde{h}$ is periodic and non-constant, then for small enough $\epsilon' > 0$, setting $h(t) = \tilde{h}(\epsilon' t)$ should result in Case I. Case I allows $\mathcal{K}(c)$ to be "more degenerate" than does (1.7): (1.7) does not imply critical points are isolated, but it implies that any point in $\mathcal{K}(c)$ is surrounded by arbitrarily small "annuli" which are disjoint with $\mathcal{K}$ (see [9]).

Case II holds, for example, if h is a constant and $n = 1$. An analogue of Case II also holds for a large class of examples for the PDE version of Theorem 1.8 (see Section 5).

Organization of Paper

The paper is organized as follows: in Section 2 are some preliminaries and technical lemmas. Section 3 contains the proof for Case I above. Section 4 contains the proof for (Case II and NOT Case I). In Section 5 we *almost* extend Theorem 1.8 to a PDE setting, and we verify its conclusion for a large class of examples.

2. Preliminaries

Here we give technical lemmas about the functionals I and I_ϵ, the manifold $\mathcal{S}$, and the analogous manifold $\mathcal{S}_\epsilon$, to be defined in a moment. Throughout the paper we assume

$$|\epsilon| < \tilde{\epsilon} \equiv \min_{\mathbf{R}} h / \sup_{\mathbf{R}} |g|. \tag{2.0}$$

Then $\inf_{\mathbf{R}}(h + \epsilon g) > 0$, so we may apply results from earlier works such as [3] and [10] (for an example of a Hamiltonian system with potential changing sign, see [11]). We assume $(V_1)-(V_3)$, $(h_1)-(h_3)$, and $(g_1)-(g_2)$ throughout. The only results in this section that require (g_3) are Proposition 2.4 and its Corollary 2.6. (V_4) is required only for Proposition 2.31.

Lemma 2.1 *Let $A > 0$ and suppose $|\epsilon| \le A$. Then I_ϵ, I'_ϵ, and I''_ϵ are bounded on bounded subsets of E, independently of ϵ.*

Proof: this is proven in [12] for $\epsilon = 0$ and is trivial to modify.

Closely related to Lemma 2.1 is the following:

Lemma 2.2 *$I_\epsilon \to I$ and $I'_\epsilon \to I'$ as $\epsilon \to 0$ uniformly on bounded subsets of E.*

Proof for $I'_\epsilon \to I'$: Let $B > 0$. By $(V_1)-(V_2)$, there exists $B_2 > 0$ such that $|V'(q)| \le B_2|q|$ if $|q| \le B$. If $\|u\| \le B$, then $\|u\|_{L^\infty(\mathbf{R})} \le B$ ([12]) and

$$\|I'_\epsilon(u) - I'(u)\| = \sup_{\|w\|=1} (I'_\epsilon(u) - I'(u))w = |\epsilon| \sup_{\|w\|=1} \int gV'(u)\cdot w \le \tag{2.3}$$

$$\le |\epsilon|(\max|g|)B_2 \sup_{\|w\|=1} \int |u||w| \le |\epsilon|(\max|g|)B_2\|u\|\|w\| \le |\epsilon|(\max|g|)B_2 B.$$

A *Palais-Smale sequence* of I is a sequence $(u_m) \subset E$ with $I'(u_m) \to 0$ and $I(u_m)$ convergent as $m \to \infty$. A functional I satisfies the *Palais-Smale condition* if any Palais-Smale sequence is precompact. Here, I does not satisfy the Palais-Smale condition, even when the $\mathbf{Z}$-symmetry of I is taken into account (see [3]). It is possible, however, to describe Palais-Smale sequences of I_ϵ (and I) via the following proposition. Assume from now on for simplicity that h is 1-periodic. For $p \in \mathbf{Z}$, define the translation operator $\tau_p : E \to E$ by $\tau_p u(t) = u(t-p)$, so $\tau_p u$ is u shifted p units to the right. Then, defining $\mathcal{K}_\epsilon = \{u \in E \mid I'_\epsilon(u) = 0\}$,

PROPOSITION 2.4 *Suppose g satisfies (g_3). Let $(u_m) \subset E$ be such that $I_\epsilon(u_m) \to b > 0$ and $I'(u_m) \to 0$. Then there exists $w_0 \in \mathcal{K}_\epsilon$ (possibly equal to 0), $l \in \mathbf{N} \cup \{0\}$, $v_1, \ldots, v_l \in \mathcal{K} \setminus \{0\}$, a subsequence of (u_m) and corresponding sequences $(k^i_m) \subset \mathbf{Z}^n$ for $1 \le i \le l$, such that*

$$\|u_m - (w_0 + \sum_{i=1}^{l} \tau_{k^i_m} v_i)\| \to 0 \tag{i}$$

and

$$|k^i_m| \to \infty \text{ and } |k^i_m - k^j_m| \to \infty \tag{ii}$$

as $m \to \infty$ for all i and all $i \neq j$, and

$$I_\epsilon(w_0) + \sum_{i=1}^{l} I(v_i) = b \tag{iii}$$

If $l = 0$ above then the summations are taken to be empty. The $\epsilon = 0$, periodic version of Proposition 2.4 is taken from [3], and essentially the same proof gives the above. See [13] and [14] (Proposition 4.35) for similar results.

It is well known that $(V_1) - (V_3)$ (or weaker growth conditions found in [10] or [15]) imply that for any $u \in E \setminus \{0\}$, the function $s \mapsto I(su)$ is increasing for small positive s, achieves a maximum over $\{s > 0\}$, then decreases to $-\infty$ as $s \to \infty$. Since $\frac{d}{ds} I(su) = I'(su)u$, this implies that $\mathcal{S}$ as defined in (1.4) is a manifold of codimension one, and any ray of the form $\{su \mid s > 0\}$ for $u \neq 0$ intersects $\mathcal{S}$ exactly once. Therefore, c is a lower bound on the critical values of I. Arguments from, for example, [10] imply that $c > 0$. Therefore we have the following two strong corollaries for Palais-Smale sequences of I and I_ϵ near level c:

COROLLARY 2.5 *Suppose g satisfies (g_3). Let $(u_m) \subset E$ be such that $0 < \liminf I(u_m) \le \limsup I(u_m) < 2c$ and $I'(u_m) \to 0$. Then either*

$$(u_m) \text{ is precompact,} \tag{i}$$

or there exists a subsequence of (u_m) (also denoted (u_m)), a sequence $(p_m) \subset \mathbf{Z}$ with $|p_m| \to \infty$, and $\bar{v} \in \mathcal{K} \setminus \{0\}$ with

$$\|u_m - \tau_{p_m} \bar{v}\| \tag{ii}$$

as $m \to \infty$.

Proof: Apply Proposition 2.4 with $\epsilon = 0$. By Proposition 2.4(iii), either $w_0 \neq 0$ and $l = 0$, or $w_0 = 0$ and $l = 1$. The first case gives Corollary 2.5(i), and the second case gives Corollary 2.5(ii).

COROLLARY 2.6 *Suppose g satisfies (g_3). Let $(u_m) \subset E$ be such that $0 < \liminf I_\epsilon(u_m) \le \limsup I_\epsilon(u_m) < 2c$ and $I'_\epsilon(u_m) \to 0$. Then I_ϵ has a nonzero critical point or Corollary 2.5 case (ii) holds.*

Proof: if I_ϵ has no nonzero critical point, then Proposition 2.4 holds with $w_0 = 0$. By Proposition 2.4(iii), $l = 1$.

Define the manifold $\mathcal{S}_\epsilon$, similar to $\mathcal{S}$, by

$$\mathcal{S}_\epsilon = \{u \mid I'_\epsilon(u)u = 0,\ u \neq 0\}. \tag{2.7}$$

(2.0) implies that $\mathcal{S}_\epsilon$ has a similar manifold structure to $\mathcal{S}$. We will frequently use the following fact:

LEMMA 2.8 *Let $B > 0$. Then the set $\mathcal{S}_\epsilon \cap \{u \mid I_\epsilon(u) \leq B\}$ is a bounded subset of E, with bounds independent of ϵ.*

The proof is contained in the proof that Palais-Smale sequences of I_ϵ are bounded, as found in, for example, [4].

Let $\tilde{\epsilon}$ be as in (2.0) and let $|\epsilon'| < \tilde{\epsilon}$. We claim there exists $r_0 = r_0(\epsilon') > 0$ with the property that that if $|\epsilon| \leq \epsilon'$, then

$$\|u\| > r_0 \tag{2.9}$$

for all $u \in \mathcal{S}_\epsilon$. Proof: let $|\epsilon'| < \tilde{\epsilon}$ and $A = \min_{\mathbf{R}} h - \epsilon' \max_{\mathbf{R}} |g| > 0$. By $(V_1) - (V_2)$, there exists $r_0 > 0$ such that if $|q| \leq r_0$, then $V'(q)q < |q|^2/(2A(p+1))$. Let $|\epsilon| \leq \epsilon'$. Suppose $u \in \mathcal{S}_\epsilon$ and $\|u\|_{L^\infty(\mathbf{R})} \leq r_0$. Then

$$\begin{aligned} 0 = I'_\epsilon(u)u = \|u\|^2 - \int_{\mathbf{R}} (h + \epsilon g)V'(u)u\,dt \geq \|u\|^2 - (\min_{\mathbf{R}} h - \epsilon' \max_{\mathbf{R}} |g|) \int_{\mathbf{R}} V'(u)u \geq \\ \geq \|u\|^2 - A(p+1) \int_{\mathbf{R}} |q|^2/(2A(p+1)) \geq \|u\|^2 - \frac{1}{2}\|u\|^2 > 0. \end{aligned} \tag{2.10}$$

This is impossible. Thus if $u \in \mathcal{S}_\epsilon$, then $\|u\| \geq \|u\|_{L^\infty(\mathbf{R})} \geq r_0$ (see [12]).

Recall that for $u \neq 0$, the function $g(s) = I(su)$ is increasing for small $s > 0$, attains a maximum, then decreases to infinity. The following lemma, from [12], estimates the change in slope of g near its maximum, which occurs at $s = 1$ if $u \in \mathcal{S}$. The strong growth condition (V_3), as opposed to weaker versions found in [15] or [3], is required for the proof.

LEMMA 2.11 *Let $u \in \mathcal{S}$ and define $g(s) = I(su)$ for $s \geq 0$. Let p be as in (V_3), and assume without loss of generality that $p \leq 2$. Then*

$$s \geq 1 \Rightarrow g'(s) \leq -\frac{1}{4}(p-1)(s-1)\|u\|^2 \tag{i}$$

and

$$\frac{1}{2} \leq s \leq 1 \Rightarrow g'(s) \geq \frac{1}{4}(p-1)(1-s)\|u\|^2. \tag{ii}$$

We will need to "normalize" functions in $E\backslash\{0\}$ to obtain functions in $\mathcal{S}_\epsilon$, that is, project functions onto the manifold $\mathcal{S}_\epsilon$. Since $\mathcal{S}_\epsilon$ is not a sphere in $W^{1,2}(\mathbf{R})$, this requires some estimates. Define $\lambda_\epsilon : E\backslash\{0\} \to \mathbf{R}^+$ by

$$\lambda_\epsilon(u) = s : s > 0, su \in \mathcal{S}_\epsilon. \tag{2.12}$$

As in [10], in an almost identical setting, λ_ϵ is a continuous function. Define $\mathcal{N}_\epsilon : E \setminus \{0\} \to \mathcal{S}_\epsilon$ by

$$\mathcal{N}_\epsilon(u) = \lambda_\epsilon u. \tag{2.13}$$

$\mathcal{N}_\epsilon$ "normalizes" functions so that they are in $\mathcal{S}_\epsilon$. The following lemma states that if ϵ is small, and $u \in \mathcal{S} \cap \{I = c\}$, then u must be close to $\mathcal{N}_\epsilon(u)$:

Lemma 2.14 *Let $\delta > 0$. There exists $\bar{\epsilon} > 0$ such that if $|\epsilon| \leq \bar{\epsilon}$, then*

$$\|u - \mathcal{N}_\epsilon(u)\| < \delta \tag{2.15}$$

for all $u \in \mathcal{S} \cap \{I = c\}$.

Proof: By Lemma 2.8, there exists $B > 0$ such that $\|u\| \leq B$ for all $u \in \mathcal{S} \cap \{I = c\}$. Assume that $B > 2\delta$. Let $\tilde{\epsilon} > 0$ be from (2.0), and let $r_0 = r_0(\tilde{\epsilon}/2)$ be as in (2.9). By Lemma 2.2, there exists $\bar{\epsilon} \in (0, \tilde{\epsilon}/2)$ such that if $|\epsilon| \leq \bar{\epsilon}$, then

$$|I'(su)u - I'_\epsilon(su)u| < \frac{1}{8}(p-1)\frac{\delta}{B}r_0^2 \tag{2.16}$$

for all u with $\|u\| \leq 2B$ and $|s| \leq 2$. Let $u \in \mathcal{S}$ with $I(u) = c$. Then $\|u\| \leq B$. Define $g(s) = I(su)$. By Lemma 2.11(i) and (2.9),

$$g'(1 + \delta/B) \leq -\frac{1}{4}(p-1)\frac{\delta}{B}\|u\|^2 \leq -\frac{1}{4}(p-1)\frac{\delta}{B}r_0^2. \tag{2.17)(i}$$

Likewise, by Lemma 2.11(ii) and (2.9), since $\delta < B/2$,

$$g'(1 - \delta/B) \geq \frac{1}{4}(p-1)\frac{\delta}{B}\|u\|^2 \leq -\frac{1}{4}(p-1)\frac{\delta}{B}r_0^2. \tag{2.17)(ii}$$

Define $g_\epsilon(s) = I_\epsilon(su)$. Since $g'(s) = I'(su)u$ and $g'_\epsilon(s) = I'_\epsilon(su)u$, (2.16) implies

$$\begin{aligned} g'_\epsilon(1 + \delta/B) &\leq g'(1 + \delta/B) + |g'_\epsilon(1 + \delta/B) - g'(1 + \delta/B)| \leq \\ &\leq -\frac{1}{4}(p-1)\frac{\delta}{B}r_0^2 + \frac{1}{8}(p-1)\frac{\delta}{B}r_0^2 < 0. \end{aligned} \tag{2.18)(i}$$

Likewise,

$$\begin{aligned} g'_\epsilon(1 - \delta/B) &\geq g'(1 - \delta/B) - |g'_\epsilon(1 + \delta/B) - g'(1 + \delta/B)| \geq \\ &\geq \frac{1}{4}(p-1)\frac{\delta}{B}r_0^2 - \frac{1}{8}(p-1)\frac{\delta}{B}r_0^2 > 0. \end{aligned} \tag{2.18)(ii}$$

Therefore there exists a unique $\bar{s} \in (1 - \delta/B, 1 + \delta/B)$ with $g'_\epsilon(\bar{s}) = I'_\epsilon(\bar{s}u)u = 0$, and $\bar{s}u = \mathcal{N}_\epsilon(u) \in \mathcal{S}_\epsilon$. So

$$\|u - \mathcal{N}_\epsilon(u)\| = \|u - \bar{s}u\| = |1 - \bar{s}|\|u\| < (\delta/B)B = \delta. \tag{2.19}$$

The following proposition implies that a critical point of $I_\epsilon|_{\mathcal{S}_\epsilon}$ is actually a critical point of I_ϵ. Even more strongly, it proves that Palais-Smale sequences of $I_\epsilon|_{\mathcal{S}_\epsilon}$ are Palais-Smale sequences of I_ϵ:

Proposition 2.20 *Let $C > 0$ and $|\epsilon_0| < \tilde{\epsilon}$ from (2.0). Then there exists $\sigma > 0$ such that if $|\epsilon| \leq \epsilon_0$, then*

$$\|I'_\epsilon|_{\mathcal{S}_\epsilon}(u)\| \geq \sigma\|I'_\epsilon(u)\| \tag{2.21}$$

for all $u \in \mathcal{S}_\epsilon$ *with* $\|u\| \le C$.

Proof: for clarity, we prove the result first for $\epsilon = 0$, then extend it. $\|I'|_{\mathcal{S}}(u)\|$ can be defined as

$$\|I'|_{\mathcal{S}}(u)\| = \sup\{I'(u)w/\|w\| \mid w \text{ is tangent to } \mathcal{S} \text{ at } u\}. \tag{2.22}$$

Define $J(u) = I'(u)u = \|u\|^2 - \int_{\mathbf{R}} hV'(u) \cdot u$. Then $\mathcal{S}$ is the level set $\{J = 0\} \setminus \{0\}$. Let ∇I and ∇J denote the gradients of I and J respectively, that is, $(\nabla I(u), w) = I'(u)w$ and $(\nabla J(u), w) = J'(u)w$ for all $u, w \in E$. For vectors $\vec{a}$ and $\vec{b} \neq 0$ in a Hilbert space, define $Proj_{\vec{b}}\vec{a}$, the projection of $\vec{a}$ onto $\vec{b}$, by $Proj_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}$, and $Perp_{\vec{b}}\vec{a}$, the projection of $\vec{a}$ onto the orthogonal complement of $\vec{b}$, by $Perp_{\vec{b}}\vec{a} = \vec{a} - Proj_{\vec{b}}\vec{a}$. Then $Perp_{\nabla J(u)}\nabla I(u)$ is tangent to $\mathcal{S}$ at u. Therefore, for any $u \in \mathcal{S}$,

$$\begin{aligned} \|I'|_{\mathcal{S}}(u)\| &\ge (I'(u)Perp_{\nabla J(u)}\nabla I(u))/\|Perp_{\nabla J(u)}\nabla I(u)\| \ge \\ &\ge (\nabla I(u), Perp_{\nabla J(u)}\nabla I(u))/\|\nabla I(u)\|, \\ \|I'|_{\mathcal{S}}(u)\|/\|I'(u)\| &\ge (\nabla I(u), Perp_{\nabla J(u)}\nabla I(u))/\|\nabla I(u)\|^2. \end{aligned} \tag{2.23}$$

We will estimate the last quantity and find a positive lower bound for it.

$$\begin{aligned} (\nabla I(u), Perp_{\nabla J(u)}\nabla I(u))/\|I'(u)\|^2 &= \left((\nabla I(u), \nabla I(u)) - \frac{(\nabla I(u), \nabla J(u))^2}{\|\nabla J(u)\|^2}\right)/\|I'(u)\|^2 = \\ &= \frac{1}{\|\nabla J(u)\|^2}\left(\|\nabla J(u)\|^2 - \frac{(\nabla I(u), \nabla J(u))^2}{\|\nabla I(u)\|^2}\right). \end{aligned} \tag{2.24}$$

$\nabla I(u)$ and u are orthogonal, since $(\nabla I(u), u) = I'(u)u = 0$. Therefore

$$\|\nabla J(u)\|^2 \ge \|Proj_u \nabla J(u)\|^2 + \|Proj_{\nabla I(u)}\nabla J(u)\|^2 = \frac{(\nabla J(u), u)^2}{\|u\|^2} + \frac{(\nabla J(u), \nabla I(u))^2}{\|\nabla I(u)\|^2}. \tag{2.25}$$

Applying (2.25) to (2.24),

$$(\nabla I(u), Perp_{\nabla J(u)}\nabla I(u))/\|I'(u)\|^2 \ge \frac{1}{\|\nabla J(u)\|^2}\frac{(\nabla J(u), u)^2}{\|u\|^2} = \frac{(J'(u)u)^2}{\|J'(u)\|^2\|u\|^2}. \tag{2.26}$$

(V_3) and the fact that $u \in \mathcal{S}$ give

$$\begin{aligned} J'(u)u &= 2\|u\|^2 - \int hV''(u)u \cdot u - \int hV'(u) \cdot u = \|u\|^2 - \int hV''(u)u \cdot u \le \\ &\le \|u\|^2 - p\int hV'(u) \cdot u = -(p-1)\|u\|^2. \end{aligned} \tag{2.27}$$

Now (2.23)-(2.27) yield

$$\|I'|_{\mathcal{S}}(u)\| \ge -(p-1)/\|J'(u)\|^2. \tag{2.28}$$

By Lemma 2.1, $\|J'\|$ is bounded on bounded subsets of E. Let $C_2 > 0$ be large enough so that $\|J'(u)\| < C_2$ if $\|u\| \le C$. Let $r_0 = r_0(0) > 0$ be as in (2.9). Then by (2.28),

$$\|I'|_{\mathcal{S}}(u)\|/\|I'(u)\| \ge \frac{(p-1)^2 r_0^2}{C_2^2}. \tag{2.29}$$

To extend this result to $\epsilon \neq 0$, note that by (2.9), since $|\epsilon| \leq \epsilon_0 < \tilde{\epsilon}$, there exists a positive lower bound $r_0^{\epsilon_0}$ for $\{\|w\| \mid w \in \mathcal{S}_\epsilon\}$. Also by Lemma 2.1, C_2 can be chosen so that $\|J'_\epsilon(u)\| < C_2$ whenever $\|u\| \leq C$.

Finally, one last result: any minimizing sequence of $I_\epsilon|_{\mathcal{S}_\epsilon}$ is a Palais-Smale sequence of I_ϵ. That is, define c_ϵ, similar to c, by

$$c_\epsilon = \inf_{\mathcal{S}_\epsilon} I_\epsilon. \tag{2.30}$$

Then

Proposition 2.31 *Assume* (V_4)*. Then if* $(u_m) \subset \mathcal{S}_\epsilon$ *with* $I_\epsilon(u_m) \to c_\epsilon$*, then*

$$I'_\epsilon(u_m) \to 0. \tag{2.32}$$

Proof: for clarity we do the proof for $\epsilon = 0$; the $\epsilon \neq 0$ case works the same. The proof is indirect. Suppose there exists $\delta > 0$ and subsequence of (u_m), also denoted (u_m), with $I_\epsilon(u_m) \to c_\epsilon$ and $\|I'(u_m)\| > \delta$ for all $m \geq 1$. By Lemma 2.8, (u_m) is a bounded sequence, so by Lemma 2.1, there exists $r > 0$ with $\|I'(w)\| > \delta/2$ for all $w \in B_r(u_m)$ for any m. Let $V : \mathcal{S} \to E$ be a locally Lipschitz vector field tangent to $\mathcal{S}$ and satisfying

$$I'(u)V(u) \geq \|I'|_{\mathcal{S}}(u)\|^2, \tag{2.33)(i}$$

$$\|V(u)\| \leq 2\|I'|_{\mathcal{S}}(u)\| \tag{2.33)(ii}$$

for all $u \in \mathcal{S}$. Recall that $\mathcal{S}$ is the level set $\{J = 0\}\setminus\{0\}$. Since J is not necessarily C^2, the existence of V may not be given by established theory of Hilbert manifolds (for example [16]). However, $V(u) = Perp_{\nabla J(u)}\nabla I(u)$ as in Proposition 2.20 satisfies (2.33)(i)-(ii) optimally (without the "2" in (2.33)(ii)). Also, this V is locally Lipschitz on $E \setminus \{0\}$, since ∇I and ∇J are locally Lipschitz on E and $\|\nabla J\|$ is nonzero except at 0 (see (2.27)). To that ∇J is locally Lipschitz, let $A > 0$ and $u \in E \setminus \{0\}$ with $\|u\| \leq A$. Let $w \in E$ with $\|u - w\| < \delta < 1$. By (V_1) and (V_4), there exists $C > 0$ with $|V''(q_1)q_1 - V''(q_2)q_2| \leq C|q_1 - q_2|$ and $|V'(q_1) - V'(q_2)| \leq C|q_1 - q_2|$ whenever $|q_1| + |q_2| \leq 2A + 2$. Then,

$$\begin{aligned}
\|J'(u) - J'(w)\| &= \sup_{z\in E,\ \|z\|=1} |J'(u)z - J'(w)z| = \qquad (2.34)\\
&= \sup_{z\in E,\ \|z\|=1} [|2(u-w,z) - \int_{\mathbf{R}} h(V''(u)u - V''(w)w)\cdot z - \int_{\mathbf{R}} h(V'(u) - V'(w))\cdot z|] \leq \\
&\leq 2\|u-w\| + (\max_{\mathbf{R}} |h|) \sup_{z\in E,\ \|z\|=1} (\int_{\mathbf{R}} |V''(u)u - V''(w)w||z| + \int_{\mathbf{R}} |V'(u) - V'(w)||z|) \leq \\
&\leq 2\|u-w\| + 2C(\max_{\mathbf{R}} |h|) \sup_{z\in E,\ \|z\|=1} \int_{\mathbf{R}} |u-w||z| \leq (2 + 2C\max_{\mathbf{R}} |h|)\|u-w\|
\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus, ∇J is locally Lipschitz on $E \setminus \{0\}$, so V is locally Lipschitz on $E \setminus \{0\}$.

Let η be the solution of the initial value problem $d\eta/ds = -V(\eta)$, $\eta(0,u) = u$. By Lemma 2.8, $\mathcal{S} \cap \{I \leq B\}$ is bounded for any $B > 0$. V is bounded on bounded subsets of E by (2.33)(ii) and Lemma 2.1. Since V is locally Lipschitz and I decreases along flow lines of η, it follows that η is continuous on $\mathbf{R}^+ \times \mathcal{S}$.

Returning to the indirect argument, (u_m) is a bounded sequence, so by Lemma 2.1 there exists $A > 0$ with $\|I'|_{\mathcal{S}}(y)\| \le \|I'(y)\| \le A$ for all $y \in B_r(u_m)$, for any m. Dropping the second argument of η for a moment, if $\eta(s) \in B_r(u_m)$, then $\|d\eta/ds\| = \|V(\eta(s))\| \le 2\|I'|_{\mathcal{S}}(u)\| \le 2A$. Therefore if $\eta(0) = u_m$, η requires time at least equal to $r/(2A)$ to escape $B_r(u_m)$. So $\eta(s, u_m) \in B_r(u_m)$ and $\|I'(\eta(s, u_m))\| \ge \delta/2$ for all $s \in [0, r/(2A)]$. But this implies

$$\begin{aligned} I(\eta(r/2A, u_m)) = I(\eta(0)) + \int_0^{r/2A} \frac{d}{ds} I(\eta)\, ds = I(u_m) - \int_0^{r/2A} I'(\eta)V(\eta) \le \\ \le I(u_m) - \int_0^{r/2A} \|I'|_{\mathcal{S}}(u)\|^2 \le I(u_m) - \int_0^{r/2A} \sigma^2 \|I'(u)\|^2 \le \\ \le I(u_m) - \int_0^{r/2A} \sigma^2 (\delta/2)^2 = I(u_m) - (r\sigma^2\delta^2)/(8A), \end{aligned} \tag{2.35}$$

for some $\sigma > 0$ as given by Proposition 2.20. Since $I(u_m) \to c$, $I(\eta(r/2A, u_m)) < c$ for large m. This contradicts the definition of c, since $\eta(r/2A, u_m) \in \mathcal{S}$.

Note: this result, with $\epsilon = 0$, implies that $\mathcal{S} \cap \{I = c\} = \mathcal{K}(c)$. Also, $\mathcal{K}(c)$ is nonempty: take $(u_m) \in \mathcal{S}$ with $I(u_m) \to c$, then apply Proposition 2.31 and Corollary 2.5.

3. Case I: Compact Component of Solutions

Here we prove Theorem 1.8in the case that the nonempty set $\mathcal{K}(c)$ has a compact component. (g_3) is not needed in this case. Let C be a compact, connected component of $\mathcal{K}(c)$. Let $r_0 = r_0(0) > 0$ be from (2.9). Also assume that r_0 is chosen small enough so that

$$N_{r_0}(\mathcal{K}(c)) \subset \ \{\frac{1}{2}c \le I \le \frac{3}{2}c\}. \tag{3.0}$$

This is possible by Lemmas 2.8 and 2.1. Since C is compact, $\overline{N_{r_0}(C)}$ is weakly closed. By (3.0), $\overline{N_{r_0}(C)}$ does not contain 0, so Corollary 2.5 with $\epsilon = 0$ implies that $\overline{N_{r_0}(C)} \cap \mathcal{K}(c)$ is compact.

We will use topological arguments similar to some found in [7]. $\partial N_{r_0}(C) \cap \mathcal{K}(c)$ is a closed subset of the compact set $\overline{N_{r_0}(C)} \cap \mathcal{K}(c)$, hence compact. By a separation theorem from point set topology ([17]), there exist disjoint compact sets A and A_2 with $A \cup A_2 = \overline{N_{r_0}(C)} \cap \mathcal{K}(c)$, $C \subset A$, and $\partial N_{r_0}(C) \cap \mathcal{K}(c) \subset A_2$. Since A and A_2 are compact and disjoint, they are separated by a positive distance. Let $\rho > 0$ with $\overline{N_{3\rho}(A)} \cap A_2 = \emptyset$. Also let ρ be small enough so that

$$\overline{N_{3\rho}(A)} \subset N_{r_0}(C). \tag{3.1}$$

This is possible because $A \subset \overline{N_{3\rho}(A)}$, A and $\partial N_{r_0}(C)$ are disjoint, and A is compact. Now

$$(\overline{N_{3\rho}(A)} \setminus A) \cap \mathcal{K}(c) = \emptyset. \tag{3.2}$$

This is true for the following reason: by choice of A, A_2, and ρ, $\overline{N_{3\rho}(A)} \cap \mathcal{K}(c) \subset N_{r_0}(C) \cap \mathcal{K}(c) \subset A \cup A_2$, while $\overline{N_{3\rho}(A)}$ is disjoint with A_2. Therefore, $\overline{N_{3\rho}(A)} \cap \mathcal{K}(c) \subset A$, so $(\overline{N_{3\rho}(A)} \setminus A)$ is disjoint with $\mathcal{K}(c)$.

We claim there exists $d > 0$ with

$$u \in \mathcal{S} \cap (\overline{N_{3\rho}(A)} \setminus N_\rho(A)) \Rightarrow I(u) > c + d. \tag{3.3}$$

To prove, argue indirectly: otherwise, there exist $(u_m) \subset \mathcal{S} \cap (\overline{N_{3\rho}(A)} \setminus N_\rho(A))$ with $I(u_m) \to c$. By Proposition 2.31, $I'(u_m) \to 0$. As before, (u_m) has a nonzero weak limit, so (u_m) is precompact, with a limit point in the closed set $\overline{N_{3\rho}(A)} \setminus N_\rho(A)$. This contradicts (3.2).

Let $B = \mathcal{S} \cap \overline{N_{2\rho}(A)}$. Choose $\epsilon > 0$ small enough so that

$$u \in B \Rightarrow \|u - \mathcal{N}_\epsilon(u)\| < \rho. \tag{3.4}$$

and

$$u \in B \Rightarrow |I_\epsilon(\mathcal{N}_\epsilon(u)u) - I(u)| < d/3. \tag{3.5}$$

(3.4)-(3.5) are possible by Lemma 2.14 and Lemma 2.2. Define $B_\epsilon = \{\mathcal{N}_\epsilon(u) \mid u \in B\}$. Then for all $w \in \partial_{\mathcal{S}_\epsilon} B_\epsilon$, $w = \mathcal{N}_\epsilon(u)$ for some $u \in \partial_{\mathcal{S}} B$, so (3.5) gives

$$I_\epsilon(w) = I_\epsilon(\mathcal{N}(u)) \geq I(u) - |I_\epsilon(\mathcal{N}_\epsilon(u)) - I(u)| > (c+d) - \frac{1}{3}d = c + \frac{2}{3}d. \tag{3.6}$$

Also, letting $u_0 \in A \subset \mathcal{K}(c)$ and $w_0 = \mathcal{N}_\epsilon(u_0) \in B_\epsilon$,

$$\inf_{B_\epsilon} I_\epsilon \leq I_\epsilon(w_0) = I_\epsilon(\mathcal{N}_\epsilon(u_0)) \leq I(u_0) + |I_\epsilon(\mathcal{N}_\epsilon(u_0)) - I(u_0)| < c + \frac{1}{3}d. \tag{3.7}$$

By Ekeland's Variational Principle ([18]) (or an original argument like the proof of Proposition 2.31), there exists a sequence $(w_m) \subset B_\epsilon$ with $I_\epsilon(w_m) \to \inf_{B_\epsilon} I_\epsilon$ and $I'_\epsilon|_{\mathcal{S}_\epsilon}(w_m) \to 0$ as $m \to \infty$. By Proposition 2.20, $I'_\epsilon(w_m) \to 0$. By the definition of B, and (3.4), $(w_m) \subset N_{3\rho}(A) \subset N_{r_0}(C)$. Apply Corollary 2.6 to the sequence (w_m). Since $\overline{N_{r_0}(C)}$ is weakly closed and does not contain 0, case (i) and not case (ii) of Corollary 2.5 holds, proving Theorem 1.8.

4. Case II: Isolated Critical Value

Here we prove Theorem 1.8 in the case that c is an isolated critical value of I *and* Case I does not hold. That is, c is an isolated critical value of I and $\mathcal{K}(c)$ has a non-compact component. This occurs, for example, if h is a constant and $n = 1$. Here (g_3) is essential: if h is a constant, and g is monotone and non-constant, then it is known (see [1]) that for any $\epsilon \neq 0$ (1.1) cannot have a homoclinic solution.

Throughout this section, we assume that (1.1) has no homoclinic solution, then obtain a contradiction. The idea of the proof is similar to that that of [19]. We will take a non-compact, connected subset C of $\mathcal{K}(c)$ and project it onto $\mathcal{S}_\epsilon$. By our assumption, $I_\epsilon|_{\mathcal{S}_\epsilon}$ has no critical points on $\mathcal{S}_\epsilon$. We use this assumption to deform C via a gradient vector flow on $\mathcal{S}_\epsilon$. After doing the deformation, we obtain a non-connected set. This is impossible, since C is connected and the deformation is continuous.

Define the "location" function $\mathcal{L} : E \setminus \{0\} \to \mathbf{R}$ by

$$\mathcal{L}(u) = t_0 : \int_{\mathbf{R}} \tan^{-1}(t - t_0)(|u'|^2 + |u|^2)\, dt = 0. \tag{4.0}$$

By the implicit function theorem, $\mathcal{L}$ is a continuous function on $E \setminus \{0\}$. Roughly, $\mathcal{L}(u)$ tells where on the real line a function u is concentrated. Define $\mathcal{S}_\epsilon(0) \subset \mathcal{S}_\epsilon$ by

$$\mathcal{S}_\epsilon(0) = \{w \in \mathcal{S}_\epsilon \mid \mathcal{L}(w) = 0\}, \tag{4.1}$$

and let

$$c_\epsilon(0) = \inf_{\mathcal{S}_\epsilon(0)} I_\epsilon. \tag{4.2}$$

Let $\hat{c} \in (c, 3c/2)$ and be close enough to c so that for some $\alpha > 0$,

$$(c, \hat{c} + \alpha) \text{ contains no critical values of } I. \tag{4.3}$$

We will see in a moment that for small enough $|\epsilon|$,

$$c_\epsilon(0) < \frac{3}{2}c. \tag{4.4}$$

It is standard to check that $c_\epsilon \le c$. If $c_\epsilon < c$ then Theorem 1.8 holds: by familiar arguments from, for example, the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([20]), there exists a sequence $(y_m) \in E$ with $I_\epsilon(y_m) \to c_\epsilon$ and $I'_\epsilon(y_m) \to 0$. By Corollary 2.5, (y_m) is precompact. Therefore we assume $c_\epsilon = c$. Clearly $c_\epsilon(0) \ge c_\epsilon = c$. Actually, we may assume

$$c_\epsilon(0) > c, \tag{4.5}$$

for if $c_\epsilon(0) = c$, then there exist $(w_m) \subset \mathcal{S}_\epsilon(0)$ with $I_\epsilon(w_m) \to c_\epsilon = c$. By Proposition 2.31, $I'_\epsilon(w_m) \to 0$. By Corollary 2.5, either (w_m) is precompact, or there exist $\bar{v} \in \mathcal{K}(c)$ and $(p_m) \subset \mathbf{Z}$ with $|p_m| \to \infty$ and $\|w_m - \tau_{p_m}\bar{v}\| \to 0$ as $m \to \infty$. The latter case is impossible, as then $\tau_{-p_m}w_m \to \bar{v}$, $\mathcal{L}(\tau_{-p_m}w_m) = \mathcal{L}(w_m) - p_m \to \mathcal{L}(\bar{v})$, so $|\mathcal{L}(w_m)| \to \infty$.

Let C be a non-compact, connected subset of $\mathcal{K}(c)$. Then $\mathcal{L}(C) \equiv \{\mathcal{L}(v) \mid v \in C\}$ is connected and unbounded in $\mathbf{R}$: $\mathcal{L}(C)$ is connected because C is connected and $\mathcal{L}$ is continuous. To show that $\mathcal{L}(C)$ is unbounded, let $(v_m) \subset C$ be a subset of C that is not precompact. By the argument above, $|\mathcal{L}(v_m)| \to \infty$.

Recall $\mathcal{N}_\epsilon$ from Lemma 2.14, and define $I^0_\epsilon : E \setminus \{0\} \to \mathbf{R}$ by

$$I^0_\epsilon(u) = I_\epsilon(\mathcal{N}_\epsilon(u)) = \max_{s>0} I_\epsilon(su). \tag{4.6}$$

Let $|\epsilon|$ be small enough so that for all $v \in \mathcal{K}(c)$,

$$\frac{1}{2}c < I^0_\epsilon(v) < \hat{c}. \tag{4.7}$$

This is possible by Lemma 2.14, Lemma 2.1, and the fact that $\mathcal{K}(c)$ is bounded (Lemma 2.8). Since $\mathcal{L}(C)$ is connected and unbounded, there exists $k \in \mathcal{L}(C) \cap \mathbf{Z}$ and $v \in C$ with $\mathcal{L}(v) = k$. Then, $\mathcal{N}_\epsilon(\tau_{-k}v) \in \mathcal{S}_\epsilon(0)$, so by (4.7), $c_\epsilon(0) \le I_\epsilon(\mathcal{N}_\epsilon(\tau_{-k}v)) = I^0_\epsilon(\tau_{-k}v) < \frac{3}{2}c$, proving (4.4).

Before proceeding further, we need the following lemma, which states roughly that $I_\epsilon \approx I$ for functions which are supported far away from $0 \in \mathbf{R}$:

Lemma 4.8 *There exists $\epsilon_0 = \epsilon_0(V, h, g)$ with the property that if $|\epsilon| \le \epsilon_0$, $(v_m) \subset \mathcal{K}(c)$ and $|\mathcal{L}(v_m)| \to \infty$, then*

$$I^0_\epsilon(v_m) \to c$$

as $m \to \infty$.

Proof: Let (v_m) be as above. It suffices to show that (v_m) has a subsequence along which $I^0_\epsilon(v_m) \to c$. Let $r_0 = r_0(0)$ be as in (2.9). By Lemma 2.11(ii), $I'(2v)v < -\frac{1}{4}(p-1)r_0^2$ for all $u \in \mathcal{K}(c)$. By Lemma 2.11, for small enough $|\epsilon|$, $I'_\epsilon(2u)u < -\frac{1}{8}(p-1)r_0^2$ for all $u \in \mathcal{K}(c)$. Therefore, by the behavior of $s \mapsto I(su)$,

$$I^0_\epsilon(v_m) = \max_{s\in[0,2]} I_\epsilon(sv_m), \tag{4.9}$$

for all m, and

$$\begin{aligned} |I^0_\epsilon(v_m) - c| = |I^0_\epsilon(v_m) - I(v_m)| &= |\max_{s\in[0,2]} I_\epsilon(sv_m) - \max_{s\in[0,2]} I(sv_m)| \leq \\ &\leq \max_{s\in[0,2]} |I_\epsilon(sv_m) - I(sv_m)| = |\epsilon| \max_{s\in[0,2]} |\int_{\mathbf{R}} gV(sv_m)| \leq \\ &\leq |\epsilon| \max_{s\in[0,2]} \int_{\mathbf{R}} |g|V(sv_m) \leq |\epsilon| \int_{\mathbf{R}} |g|V(2v_m). \end{aligned} \tag{4.10}$$

Let $\delta > 0$. Let $R > 0$ with $|g| < \delta$ on $\{|t| \geq R\}$. (v_m) is bounded in E (Lemma 2.8), hence in $L^\infty(\mathbf{R})$, so by $(V_1) - (V_2)$, there exists B with $V(2v_m(t)) < B|2v_m(t)|^2$ for all m and all $t \in \mathbf{R}$. Thus

$$\begin{aligned} \int_{\mathbf{R}} |g|V(2v_m)\,dt &= \int_{\{|t|>R\}} |g|V(2v_m)\,dt + \int_{\{|t|<R\}} |g|V(2v_m)\,dt \leq \\ &\leq \delta \int_{\{|t|>R\}} 4B|v_m|^2\,dt + 4B\int_{\{|t|<R\}} |g||v_m|^2\,dt \leq \\ &\leq 4B\|v_m\|^2\delta + 4B(\max_{\mathbf{R}}|g|)\int_{\{|t|<R\}} |v_m|^2\,dt. \end{aligned} \tag{4.11}$$

Apply Corollary 2.5 with $\epsilon = 0$ to (v_m). Since $|\mathcal{L}(v_m)| \to \infty$, Corollary 2.5 case (ii) applies and $v_m \to 0$ in $W^{1,2}_{loc}(\mathbf{R})$. Thus (4.10)-(4.10) yield

$$\limsup_{m\to\infty} |I^0_\epsilon(v_m) - c| \leq \delta(4B|\epsilon| \limsup_{m\to\infty} \|v_m\|^2). \tag{4.12}$$

Since δ is arbitrary, $I^0_\epsilon(v_m) \to c$.

By Lemma 4.8, we may choose $R > 0$ so that if $v \in \mathcal{K}(c)$ with $|\mathcal{L}(v)| \geq R$, then

$$I^0_\epsilon(v) < (c + c_\epsilon(0))/2 < c_\epsilon(0). \tag{4.13}$$

$\mathcal{L}(C)$ is connected and unbounded. By translating C along the real line if necessary, we may assume that there exist $z_0, z_1 \in C$ with $\mathcal{L}(z_0) = -R$ and $\mathcal{L}(z_1) = R$. "Normalize" C to obtain

$$Y_1 = \mathcal{N}_\epsilon(C) \equiv \{\mathcal{N}_\epsilon(u) \mid u \in C\}, \qquad w_0 = \mathcal{N}_\epsilon(z_0), \qquad w_1 = \mathcal{N}_\epsilon(z_1). \tag{4.14}$$

Then $w_0, w_1 \in Y_1$, $\mathcal{L}(w_0) = -R$, $\mathcal{L}(w_1) = R$, $I_\epsilon(w_0) < (c+c_\epsilon(0))/2$, $I_\epsilon(w_1) < (c+c_\epsilon(0))/2$, Y_1 is a connected subset of $\mathcal{S}_\epsilon$, and $I_\epsilon(w) < \hat{c}$ for all $w \in Y_1$ by (4.7).

Let us deform Y_1 to obtain a contradiction. Like in the proof of Proposition 2.31, let $V_\epsilon : \mathcal{S}_\epsilon \to E$ be a locally Lipschitz vector field tangent to $\mathcal{S}_\epsilon$ and satisfying

$$I'_\epsilon(u)V_\epsilon(u) \geq \|I'_\epsilon|_{\mathcal{S}_\epsilon}(u)\|^2, \tag{4.15)(i}$$

$$\|V_\epsilon(u)\| \leq 2\|I'_\epsilon|_{\mathcal{S}_\epsilon}(u)\| \tag{4.15)(ii}$$

for all $u \in \mathcal{S}_\epsilon$. Let $\varphi : E \to [0,1]$ be a locally Lipschitz cutoff function with $\varphi \equiv 0$ on $\{I_\epsilon \le (c + c_\epsilon(0))/2\}$ and $\varphi \equiv 1$ on $\{I_\epsilon \ge c/3 + 2c_\epsilon(0)/3\}$. As in the proof of Proposition 2.31, the initial value problem $d\eta/dt = -\varphi(\eta)V_\epsilon(\eta)$, $\eta(0,w) = w$ has a continuous solution $\eta : \mathbf{R}^+ \times \mathcal{S}_\epsilon \to \mathcal{S}_\epsilon$.

We have been assuming that there is no critical point w of I_ϵ with $c \le I_\epsilon(w) \le 2c$. Therefore, by Corollary 2.6, Proposition 2.20, and the definition of $\hat{c}$, there exists $\delta > 0$ such that $\|I'_\epsilon|_{\mathcal{S}_\epsilon}\| > \delta$ for all $w \in \mathcal{S}_\epsilon \cap \{(c + c_\epsilon(0))/2 \le I_\epsilon \le \hat{c}\}$. Define

$$Y_2 = \eta(2c/\delta^2, Y_1) \equiv \{\eta(2c/\delta^2, w) \mid w \in Y_1\}. \tag{4.16}$$

Y_2 is connected, since Y_1 is connected and η is continuous. Thus $\mathcal{L}(Y_2)$ is connected. $w_0, w_1 \in Y_2$, since $w_0, w_1 \in Y$, $I_\epsilon(w_0) < (c + c_\epsilon(0))/2$, $I_\epsilon(w_1) < (c + c_\epsilon(0))/2$, and $\varphi \equiv 0$ on $\{I_\epsilon \le (c + c_\epsilon(0))/2\}$. Since $\mathcal{L}(w_0) < 0 < \mathcal{L}(w_1)$ and $\mathcal{L}(Y_2)$ is connected, there exists $\tilde{w} \in Y_2$ with $\mathcal{L}(\tilde{w}) = 0$. Since $\tilde{w} \in \mathcal{S}_\epsilon(0)$, $I_\epsilon(\tilde{w}) \ge c_\epsilon(0)$ by definition of $c_\epsilon(0)$.

This is impossible because for all $w \in Y_2$, $I_\epsilon(w) \le c/3 + 2c_\epsilon(0)/3 < c_\epsilon(0)$. The proof of this is indirect. Suppose $w \in Y_2$ with $I_\epsilon(w) \ge c/3 + 2c_\epsilon(0)/3$. w has the form $w = \eta(2c/\delta^2, u)$ for some $u \in Y_1$. For all $s \in [0, 2c/\delta^2]$, $I_\epsilon(\eta(s,u)) \ge c/3 + 2c_\epsilon(0)/3$. By (4.7), $I_\epsilon(u) < \hat{c}$. The choice of δ and the definition of φ imply that $\|I'_\epsilon(\eta(s,u))\| \ge \delta$ and $\varphi(\eta(s,u)) = 1$ for all $s \in [0, 2c/\delta^2]$. Therefore,

$$\begin{aligned} I_\epsilon(\eta(2c/\delta^2, u)) = I_\epsilon(u) + \int_0^{2c/\delta^2} \frac{d}{ds} I_\epsilon(\eta(s,u))\, ds = I_\epsilon(u) - \int_0^{2c/\delta^2} I'_\epsilon(\eta(s,u)) V(\eta(s,u))\, ds \le \\ \le I_\epsilon(u) - \int_0^{2c/\delta^2} \delta^2 \le c_\epsilon(0) - (2c/\delta^2)\delta^2 = c_\epsilon(0) - 2c < \hat{c} - 2c < -c/2 < 0. \end{aligned} \tag{4.17}$$

This is impossible, since $I_\epsilon > 0$ on $\mathcal{S}_\epsilon$. Theorem 1.8, [Case II and not Case I] is proven.

5. A PDE Analogue

Here we pose the obvious PDE analogue of Theorem 1.8. Most of the steps of the proof can be followed as before. However, the proof fails for "Case II." Differences between the topology of $\mathbf{R}$ and of $\mathbf{R}^n$ $(n > 1)$ are to blame. Despite this failure, we can verify the conlusion of the Theorem 1.8 analogue for a large class of autonomous PDE.

Let $n \ge 2$ and consider the equation

$$-\Delta u + u = h(x) f(u), \tag{5.0}$$

where f and h satisfy

(f_1) $f \in C^1(\mathbf{R}, \mathbf{R})$

(f_2) $f(0) = 0 = f'(0)$

(f_3) there exists $p > 1$ such that $f'(q)q^2 \ge p f(q) q > 0$ for all $q \in \mathbf{R} \setminus \{0\}$

(f_4) there exist $a_1, a_2, s > 1$ with $|f'(q)| \le a_1 + a_2 |q|^{s-1}$ for all $q \in \mathbf{R}$, with $s < (n+2)/(n-2)$ if $n \ge 3$

(f_5) there exist $C, r > 0$ with

$$|f'(q_1)q_1 - f'(q_2)q_2| \le C(1 + |q_1|^r + |q_2|^r)|q_1 - q_2|$$

for all $q_1, q_2 \in \mathbf{R}$, with $r = 4/(n-2)$ if $n \geq 3$

(h_1) $h \in C^1(\mathbf{R}^n, \mathbf{R})$

(h_2) $h(x) > 0$ for all $x \in \mathbf{R}^n$

(h_3) h is periodic in $x_1, x_2, \ldots, x_n$.

These are essentially the same conditions as those for Theorem 1.8, except for the additional subcritical-growth condition (f_4), which is not needed in the $n = 1$ case. (f_5) is an analogue of (V_4) from Theorem 1.8. $(f_1) - (f_5)$ hold if, for example, $f(q) = |q|^{s-1}q$ with s as in (f_4). Equation (5.0) or more general versions have been the subject of much study, including [21] and [9].

Define the energy functional $I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^n} hF(u)$ on $W^{1,2}(\mathbf{R}^n)$, where F is the primitive or antiderivative of f. Critical points of I correspond to "homoclinic-type" solutions of (5.0), that is, solutions u with $|u(x)| + |\nabla u(x)| \to 0$ as $|x| \to \infty$. We may define $\mathcal{S}$, c, and $\mathcal{K}(c)$ exactly as before in the Introduction. Since analogues for all the results for Section 2 hold here, $\mathcal{K}(c)$ is nonempty.

If $\mathcal{K}(c)$ has a compact component (Case I), then suppose g satisfies

(g_1) $g \in C^1(\mathbf{R}^n, \mathbf{R})$ and

(g_2) $\sup_{x \in \mathbf{R}^n} |g(x)| < \infty$.

The results of Section 2 can be proven easily (with the exception of those which involve (f_5); we will verify these in a moment). Then the arguments of Section 3 can be applied to show that for small enough $|\epsilon|$, the equation

$$-\Delta u + u = (h(x) + \epsilon g(x))f(u), \tag{5.1}$$

has a nonzero homoclinic-type solution. As before, (g_3) is not needed.

In the case that c is an isolated critical value of I and Case I does not hold (Case II and *not* Case I), things are not so clear. Before explaining why the the $n = 1$ proof does not generalize, let us look at a class of h's for which the perturbed equation (5.1) *does* have a homoclinic-type solution. Namely,

Theorem 5.2 *If $h \equiv$ constant > 0, f satisfies $(f_1) - (f_5)$, c is an isolated critical value of I and g satisfies $(g_1) - (g_2)$ and*

(g_3) $g(x) \to 0$ as $|x| \to \infty$,

then there exists $\epsilon_0 > 0$ with the property that if $|\epsilon| \leq \epsilon_0$, then (5.1) *has a homoclinic solution.*

Proof: We will be sketchy at times because most of the work has been done in Sections 2 and 4. All of the results in Section 2 have analogues here. Most are easy to prove, but one should be noted. We need to verify the fact that J' is locally Lipschitz on $E \equiv W^{1,2}(\mathbf{R}^n)$, where $J(u) = I'(u)u$. This is needed for Proposition 2.31, and also for arguments like those in Section 4 which employ a gradient vector field for I_ϵ constrained to $\mathcal{S}_\epsilon$. We will look at the $n \geq 3$ case; the $n = 2$ case is similar.

Let $u, w \in E$. We must bound $\|J'(u) - J'(w)\|/\|u - w\|$ in terms of $\|u\|$ and $\|w\|$: $\|J'(u) - J'(w)\| = \sup_{\|z\|=1} |(J'(u) - J'(w))z|$, so let $\|z\| = 1$. Then

$$(J'(u) - J'(w))z = 2(u - w, z) - \int_{\mathbf{R}^n} (f'(u)u - f'(w)w)z - \int_{\mathbf{R}^n} (f(u) - f(w)z, \tag{5.3}$$

$$|(J'(u) - J'(w))z| \leq 2\|u - w\| + \int_{\mathbf{R}^n} |f'(u)u - f'(w)w||z| - \int_{\mathbf{R}^n} |f(u) - f(w)||z|.$$

We must estimate the last two integrals. By (f_3),

$$\int_{\mathbf{R}^n} |f'(u)u - f'(w)w||z| \le C\left(\int_{\mathbf{R}^n} |u-w||z| + \int_{\mathbf{R}^n} |u|^r|u-w||z| + \int_{\mathbf{R}^n} |w|^r|u-w||z|\right). \tag{5.4}$$

The first integral above is easy to estimate via the Cauchy-Schwarz inequality. The second and third are similar, so let us look at the third. Let $\bar{n} = 2n/(n-2)$. Then $r = 4/(n-2) = 2\bar{n}/n$. $1/(\frac{n}{2}) + 1/\bar{n} + 1/\bar{n} = 1$, so by Hölder's inequality,

$$\begin{aligned}\int_{\mathbf{R}^n} |w|^r|u-w||z| &= \int_{\mathbf{R}^n} |w|^{2\bar{n}/n}|u-w||z| \le \||u|^{2\bar{n}/n}\|_{L^{n}/2(\mathbf{R}^n)}\|u-w\|_{L^{\bar{n}}(\mathbf{R}^n)}\|z\|_{L^{\bar{n}}(\mathbf{R}^n)} = \\ &= \|u\|^{2\bar{n}/n}_{L^{\bar{n}}(\mathbf{R}^n)}\|u-w\|_{L^{\bar{n}}(\mathbf{R}^n)}\|z\|_{L^{\bar{n}}(\mathbf{R}^n)} \le A_n\|u\|^{2\bar{n}/n}\|u-w\|\end{aligned} \tag{5.5}$$

for some A_n depending on n.

The last integral in (5.3) is similar. By the mean value theorem, for every $q_1, q_2 \in \mathbf{R}$ there is a ϕ between q_1 and q_2 with

$$\begin{aligned}|f(q_1) - f(q_2)| &\le |f'(\phi)||q_1 - q_2| \le (a_1 + a_2|\phi|^{s-1})|q_1 - q_2| \le B_1(1 + |\phi|^{\bar{n}-2})|q_1 - q_2| \le \\ &\le B_2(1 + |q_1|^{\bar{n}-2} + |q_2|^{\bar{n}-2})|q_1 - q_2|,\end{aligned} \tag{5.6}$$

where B_2 depends on a_1, a_2, and n. Therefore

$$\begin{aligned}\int_{\mathbf{R}^n} |f(u) - f(w)||z| &\le B_2\int_{\mathbf{R}^n} (1 + |u|^{\bar{n}-2} + |w|^{\bar{n}-2})|u-w||z| = \\ &= B_2\left(\int_{\mathbf{R}^n} |u-w||z| + \int_{\mathbf{R}^n} |u|^{\bar{n}-2}|u-w||z| + \int_{\mathbf{R}^n} |w|^{\bar{n}-2}|u-w||z|\right).\end{aligned} \tag{5.7}$$

Since $\bar{n} - 2 = 2\bar{n}/n$, the argument of (5.5) provides the estimate.

The main proof idea of Theorem 5.2 is similar to that in [19]. Let $v_0 \in \mathcal{K}(c)$. For $x \in \mathbf{R}^n$, define the translation operator $\tau_x : W^{1,2}(\mathbf{R}^n) \to W^{1,2}(\mathbf{R}^n)$ like in Section 2. Let $C = \{\tau_x v_0 \mid x \in \mathbf{R}^n\}$. Since (5.0) is autonomous, $C \subset \mathcal{K}(c)$. Define $\hat{c} \in (c, 3c/2)$ as in (4.3). Define I_ϵ and I_ϵ^0 as in (1.3) and (4.6) and let $|\epsilon|$ be small enough so that

$$u \in C \Rightarrow \frac{1}{2}c < I_\epsilon^0(u) < \hat{c}. \tag{5.8}$$

Define the "location" function $\mathcal{L}$, similar to $\mathcal{L}$ from (4.0), as follows: for $i = 1, 2, \ldots, n$, define $\mathcal{L}_i : W^{1,2}(\mathbf{R}^n)\setminus\{0\} \to \mathbf{R}$ by

$$\mathcal{L}_i(u) = t : \int_{\mathbf{R}} \tan^{-1}(x_i - t)(|\nabla u|^2 + |u|^2)\,dx = 0, \tag{5.9}$$

and $\mathcal{L}(u) = (\mathcal{L}_1(u), \ldots, \mathcal{L}_n(u))$. Define $\mathcal{S}_\epsilon$ as in (4.1), and let

$$\mathcal{S}_\epsilon(0) = \{w \in \mathcal{S} \mid \mathcal{L}(w) = 0\} \tag{5.10}$$

and

$$c_\epsilon(0) = \inf_{\mathcal{S}_\epsilon(0)} I_\epsilon. \tag{5.11}$$

As before, if $c_\epsilon(0) \le c$ we can use a multi-dimensional version of Corollary 2.6 (see [21]) to show that I_ϵ has a nonzero critical point. So assume $c_\epsilon(0) > c$. As before, for small enough ϵ, $c_\epsilon(0) < \hat{c}$.

Assume without loss of generality that $\mathcal{L}(v_0) = 0$. Let R be large enough so that

$$|x| \geq R \Rightarrow I_\epsilon^0(\tau_x v_0) < (c + c_\epsilon(0))/2. \tag{5.12}$$

Now suppose that I_ϵ has no critical values between c and $2c$. Like at the end of Section 4, there exists a continuous deformation $\eta : \mathbf{R}^+ \times \mathcal{S}_\epsilon \to \mathcal{S}_\epsilon$ and a number $T > 0$ with the properties that $\eta(0, w) = w$ for all $w \in \mathcal{S}_\epsilon$, $\eta(t, w) = w$ for all $t \geq 0$ if $w \in \mathcal{S}_\epsilon$ with $I_\epsilon(w) \leq (c + c_\epsilon(0))/2$, and $I_\epsilon(\eta(T, w)) \leq c/3 + 2c_\epsilon(0)/3$ for all $w \in \mathcal{S}_\epsilon$ with $I_\epsilon(w) \leq \hat{c}$.

Let $\mathcal{N}_\epsilon$ be as in (2.13). Define $H : [0, T] \times \overline{B_R(0)} \to \mathbf{R}^n$ by $H(t, x) = \mathcal{L}(\eta(t, \mathcal{N}_\epsilon(\tau_x v_0)))$. Now $H(0, x) = \mathcal{L}(\tau_x v_0) = x$ for all $x \in \overline{B_R(0)}$. For all $x \in \partial B_R(0)$ and all $t \in [0, T]$, $I_\epsilon^0(\tau_x v_0) < (c + c_\epsilon(0))/2$ by (5.12), so

$$H(t, x) \equiv \mathcal{L}(\eta(t, \mathcal{N}_\epsilon(\tau_x v_0))) = \mathcal{L}(\mathcal{N}_\epsilon(\tau_x v_0)) = \mathcal{L}(\tau_x v_0) = x \neq 0 \tag{5.13}$$

for $t \in [0, T]$. Therefore the Brouwer degree $d(H(t, \cdot), B_R(0), 0)$ is well-defined for all $t \in [0, T]$, and is the same for all such t. Since $H(0, x) \equiv x$, $d(H(T, \cdot), B_R(0), 0) = 1 \neq 0$. Thus there exists $x^0 \in B_R(0)$ with $H(T, x^0) \equiv \mathcal{L}(\eta(T, \tau_{x^0} v_0)) = 0$, so by the definition of $c_\epsilon(0)$,

$$I_\epsilon(\eta(T, \tau_{x^0} v_0)) \geq c_\epsilon(0). \tag{5.14}$$

This is impossible because $I_\epsilon(\tau_{x^0} v_0) < \hat{c}$ and $I_\epsilon(\eta(T, w)) \leq c/3 + 2c_\epsilon(0)/3 < c_\epsilon(0)$ for all $w \in \mathcal{S}_\epsilon$ with $I_\epsilon(w) \leq \hat{c}$. Theorem 5.2 is proven.

An Example of Theorem 5.2

The hypotheses of Theorem 5.2 can be verified for the following example: $h \equiv 1$, and in addition to $(f_1)-(f_5)$, (5.0) has a unique (modulo translation) positive homoclinic-type solution. There are many known examples of this uniqueness ([22]). In this case, it is straightforward to prove that c is an isolated critical value of I, using elliptic estimates and the maximum principle.

Despite Theorem 5.2, in general in the case that c is an isolated critical value of I and $\mathcal{K}(c)$ has a non-compact component, we cannot hope to make the arguments of Section 4 work. This is why: in Section 4, we found a non-compact, connected set of solutions $C \subset \mathcal{K}(c)$ to the unperturbed problem. We showed that $\mathcal{L}(C)$ must be a connected, unbounded set. By translating C, we were able to find a connected subset C of $\mathcal{K}(c)$ such that $\mathcal{L}(C)$ contained an arbitrarily large neighborhood of $0 \in \mathbf{R}$ (the interval $[-R, R]$ in (4.13)). This was the key to setting up the connectedness/deformation argument.

Working in more than one dimension, if C is a connected, noncompact subset of $\mathcal{K}(c)$, then $\mathcal{L}(C) \subset \mathbf{R}^n$ is (still) a connected, unbounded set. But an arbitrary unbounded, connected subset of R^n ($n \geq 2$) may have empty interior, so the argument which succeeded in Theorem 5.2 is impossible.

Open Questions

There are many open questions associated with (1.1) and similar problems. A dearth of counterexamples (functions h and V in (1.0) for which it is known that no homoclinic-type solution exists) makes such questions easy to pose. For example, we saw above that Theorem 1.8 cannot easily be extended to more than one dimension, and indeed whether the theorem is true in this case is unknown. It is unknown whether Case I or Case II are necessary for the result of Theorem 1.0. It is not even known whether $|\epsilon|$ has to be small!

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