A Singularly Perturbed Elliptic Partial Differential Equation with an Almost Periodic Term

Gregory S. Spradlin
United States Military Academy
West Point, New York, USA

1. Introduction

In [STT], a Hamiltonian system of the form

$$-u'' + u = h(t)\nabla F(u) \tag{1.0}$$

was studied, where h is an almost periodic (defined in a moment) function, and $F : \mathbf{R}^n \to \mathbf{R}$ a "superquadratic" potential. That is, $F(q)$ behaves like q to a power greater than 2, with $F(q)/|q|^2 \to 0$ as $|q| \to 0$ and $F(q)/|q|^2 \to \infty$ as $|q| \to \infty$. For example, $F(q) = |q|^{p-1}q$ with $p > 1$ would qualify. The authors found that (1.0) must have a nonzero solution homoclinic to zero. Since this result, many papers (see [CMN], [R1], and [ACM], for example) have been written concerning Hamiltonian systems with almost periodic terms.

As we will see, it is natural to extend the definition of almost periodic to functions on $\mathbf{R}^n$, $n > 1$, or even to more general topological groups. Thus one can write a PDE version of (1.0),

$$-\Delta u + u = h(x)f(u), \tag{1.1}$$

wherein h is almost periodic and the primitive F of f satisfies appropriate superquadraticity and growth conditions. Then one may ask, does (1.1) have a "homoclinic-type" solution? That is, is there a nonzero solution u with $|u(x)| + |\nabla u(x)| \to 0$ as $|x| \to \infty$? Here we take a step towards answering in the affirmative.

Let us define an almost periodic function on $\mathbf{R}^n$ ($\mathbf{R}$ is a special case, and defining an a.p. function on other topological groups is an obvious generalization). First, a set $\mathcal{A} \subset \mathbf{R}^n$ is *relatively dense* if there exists $L > 0$ such that for every $x \in \mathbf{R}^n$, there exists $y \in \mathcal{A}$ with $|x - y| < L$. Next, for $\epsilon > 0$, $\vec{v} \in \mathbf{R}^n$, and $h : \mathbf{R}^n \to \mathbf{R}$, we say $\vec{v}$ is an *ϵ-almost period* of h if for all $x \in \mathbf{R}^n$, $|h(x + \vec{v}) - h(x)| < \epsilon$. Finally, h is defined to be *almost periodic* if for every $\epsilon > 0$, there exists a relatively dense set $\mathcal{A} \equiv \mathcal{A}(\epsilon) \subset \mathbf{R}^n$ such that for all $a \in \mathcal{A}$, a is an ϵ-almost period of h. For properties of almost periodic functions (many properties of a.p. functions on $\mathbf{R}$ extend to a.p. functions on $\mathbf{R}^n$), see [Be], [Bo], [C], [Z].

We will look at an equation similar to (1.1), of the form

$$-\epsilon^2 \Delta\tilde{u} + V(x)\tilde{u} = f(\tilde{u}) \tag{1.2}$$

on $\mathbf{R}^n$. Equations like (1.2) arise in the study of the nonlinear Schrödinger equation and have been the subject of much study recently (see [R2], [FdP1-3], [Li] and the references therein). We will assume that V and f satisfy the following conditions:

(V_1) $V \in C^1(\mathbf{R}^n, \mathbf{R})$,
(V_2) $0 < V_- \le \inf_{\mathbf{R}^n} V \le \sup_{\mathbf{R}^n} V \le V^+ < \infty$
(V_3) V is almost periodic.
(f_1) $f \in C^1(\mathbf{R}^+, \mathbf{R})$
(f_2) $f'(0) = 0 = f(0)$.
(f_3) There exist $A, s > 0$ such that $|f'(q)| \le A(1 + |q|^{s-1})$ for all $q \ge 0$. If $n \ge 3$, then $s < 4/(n-2)$.
(f_4) For some $\mu > 2$, $0 < \mu F(q) \le f(q)q$ for all $q > 0$, where $F(\xi) \equiv \int_0^\xi f(t)\,dt$.
(f_5) The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.
(f_6) For every $a > 0$, the equation $-\Delta u + au = f(u)$ has a unique (modulo translation) positive solution.

$(f_1) - (f_6)$ are satisfied if, for example, $f(q) = q^s$ with s as in (f_3) (for verification of (f_6), see [GNN1-2] and [Y]). $(f_1) - (f_4)$ give the "superquadratic" character of f. (f_5) is a useful convexity assumption found in many papers such as [R2], [WZ], and [FdP1-3]. In [FdP2] it was shown that, under conditions weaker than $(V_1) - (V_2)$ and $(f_1) - (f_6)$, if V has a "topologically nontrivial" set of critical points, then for small enough ϵ, (1.2) has a positive solution u with $|u(x)| + |\nabla u(x)| \to 0$ as $|x| \to \infty$. A topologically nontrivial set of critical points is a compact set of critical points of V obtained by a topological "linking." Such a set of critical points has the property that if V is perturbed a little in C^1, there are still critical points nearby. Let us define such a linking: suppose there exists a bounded open set Λ and closed sets B_0 and B with B nonempty and connected, and $B_0 \subset B \subset \Lambda$. Let Γ be the class of all continuous functions $\phi : B \to \Lambda$ with the property that $\phi(y) = y$ for all $y \in B_0$. Define the minimax value c as

$$c = \inf_{\phi \in \Gamma} \sup_{y \in B} V(\phi(y)), \tag{1.3}$$

and assume

$$B \ne \emptyset \Rightarrow \sup_{B_0} V < c \tag{1.4}$$

and

$$\text{For all } \phi \in \Gamma,\ \phi(B) \cap \{y \in \Lambda \mid V(y) \ge c\} \ne \emptyset. \tag{1.5}$$

In the language of the calculus of variations, the sets B_0, B, and $\{V \ge c\}$ *link* in Γ. Also assume

$$\text{For all } y \in \partial\Lambda,\ \partial_\tau V(y) \ne 0, \tag{1.6}$$

where ∂_τ denotes tangential derivative. For example, such B_0, B, Λ exist if V has a strict local maximum or minimum; that is, if there exists a bounded open set O with $\max_{\partial O} V < \sup_O V$ or $\inf_{\partial O} V < \inf_O V$. Another example is a "saddle point". For example, $(0,0)$ is a topologically stable critical point of $V(x_1, x_2) = x_1^2 - x_2^2$. In [FdP2] it was shown, under weaker conditions than $(V_1) - (V_2)$, $(f_1) - (f_6)$, and (1.3)-(1.6), that for small enough ϵ, (1.2) has a positive homoclinic-type solution.

We would like to show that (1.2) has a nontrivial homoclinic-type solution for small enough ϵ if we do not assume (1.3)-(1.6) but assume instead that V is almost periodic. It is easy to see that an almost periodic function on $\mathbf{R}$ is either constant or has an infinite number of topologically nontrivial local maxima and minima. However, an a.p. function on $\mathbf{R}^n$ need not have a topologically nontrivial critical point. For example, consider $V(x_1, x_2) = 2 + \sin x_1$. If this V occured in (1.2), then the variational arguments can be made one-dimensional, so the equation with this V would not be too challenging. For a more interesting example, let $g : \mathbf{R} \to \mathbf{R}$ be almost periodic, and define $V(x) = 2 + \sin(x_1 - g(x_2))$. Then V is almost periodic, nonconstant and has no topologically stable critical points. Indeed, an a.p. function of several variables need not have any local minimum at all, let alone a topologically nontrivial one ([S]). So the work of [FdP] will not give the desired result.

We will prove the following:

THEOREM 1.7 *Let V and f satisfy $(V_1) - (V_3)$ and $(f_1) - (f_6)$, and assume as well that V satisfies one of the following three cases:*

(I) $V \equiv$ constant,

(II) there exists a bounded open set $\Lambda \subset \mathbf{R}^n$ and closed sets $B_0 \subset B \subset \Lambda$ satisfying (1.3)-(1.6).

(III) there exists an open set $O \subset \mathbf{R}^n$ with $\inf_{\partial O} V > \inf_O V$ and $\vec{v} \in \mathbf{R}^n \setminus \{0\}$ with

$$\sup\{\vec{u} \cdot x \mid x \in O,\ \vec{u} \perp \vec{v},\ \|\vec{u}\| = 1\} < \infty.$$

Then there exists $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$, then (1.2) has a positive homoclinic-type solution $\tilde{u}_\epsilon$. $\tilde{u}_\epsilon$ has exactly one local maximum (hence, global maximum) point $z_\epsilon \in \mathbf{R}^n$. Further, there exist $\alpha, \beta > 0$ with $\tilde{u}_\epsilon(z) \le \alpha \exp(-\frac{\beta}{\epsilon}|z - z_\epsilon|)$ for $\epsilon \le \epsilon_0$. In Case III, $V(z_\epsilon) \to \inf_O V$ as $\epsilon \to 0$.

More detailed conclusions for Case II are given in [FdP2]. O above can be thought of as a "tube" that is bounded in all but possibly one direction in $\mathbf{R}^n$. While the above result is strong, it is especially interesting because one of Cases I-III is automatically satisfied when $n = 2$:

THEOREM 1.8 *If $V : \mathbf{R}^2 \to \mathbf{R}$ is continuous and almost periodic, then V satisfies one of Cases (1.7)I-III.*

Note: Theorem 1.8 does not hold for $n \ge 3$; consider $V(x_1, x_2, x_3) = 2 + \sin x_1$. Putting Theorems 1.7 and 1.8 together yields:

COROLLARY 1.9 *Let $n = 2$ and let V and f satisfy $(V_1) - (V_3)$ and $(f_1) - (f_6)$. Then there exists $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$, then (1.2) has a nontrivial homoclinic-type solution u.*

Let us compare equations (1.1) and (1.2). In (1.2), the coefficient function V is placed differently, in order to take advantage of recent results for equations of the same general form as (1.2). If V were moved in front of $f(u)$, the solution techniques would be essentially the same, so this difference is not very important. Knowing that one of Cases I-III hold (or simply that $n = 2$) is essential for this proof. The most troubling restriction, however, is the presence of ϵ^2 in front of Δu. This is equivalent to dilating V in the plane. Despite these limitations, Theorem 1.7 is not easy to prove and seems markedly different from anything in the literature.

Proof of $n = 1$ result

Before outlining the proof of Theorem 1.3, let us outline the proof of [STT]'s result to show why it cannot be applied here. That proof fails because of differences in the topology of $\mathbf{R}$ and $\mathbf{R}^n$ for $n > 1$. It is because of those differences that we impose the limitations described above.

For equation (1.0), let $E = W^{1,2}(\mathbf{R})$ and define $I \in C^1(E, \mathbf{R})$ by $I(u) = \frac{1}{2}\|u\|^2 - \int h(t)F(u)\,dt$. Then critical points of I are homoclinic solutions of (1.0). Using variational mountain-pass techniques, [STT] construct a sequence $(u_m) \subset E$ with $I'(u_m) \to 0$, $I(u_m) \to b > 0$, and $\|u_m - u_{m-1}\| \to 0$ as $m \to \infty$. Then they show that there exist $\delta > 0$ and a sequence $(t_m) \subset \mathbf{R}$ with $|u_m(t_m)| > \delta$ for all m and $|t_m - t_{m-1}| \to 0$. They use (u_m) to construct a Palais-Smale sequence which "concentrates" near some value of t and therefore has a nonzero weak limit. If (t_m) has a bounded subsequence then this occurs already. So assume $|t_m| \to \infty$. For $s \in \mathbf{R}$, let τ_s denote the translation operator given by $\tau_s(u)(t) = u(t-s)$. Since $|t_m - t_{m-1}| \to 0$, for each $l = 1, 2, 3, \ldots$ there exists a $1/l$-period s_l of h and $m_l \in \mathbf{N}$ with $|m_l - s_l| < 1/l$. Define $v_l = \tau_{-s_l} u_{m_l}$. It is straightforward to show that $I'(v_l) \to 0$. The u_m's are uniformly continuous independently of m, so it follows that $\liminf_{l \to \infty} |v_l(0)| > 0$. Now it is standard to check that along a subsequence, (v_l) converges weakly to a nonzero critical point of I.

For a multidimensional version of (1.0), such as (1.1), it is still possible to define a functional I corresponding to (1.1), and find a Palais-Smale sequence (u_m) with the above properties, with $(x_m) \subset \mathbf{R}^n$ having similar properties to (t_m). But even though $|x_m - x_{m-1}| \to 0$, it is no longer the case that x_m must pass arbitrarily close to any ϵ-periods of h for ϵ small. Here the attempt to copy [STT]'s proof breaks down.

Variational Framework and Plan of Proof

By a change of coordinates, we can recast (1.2) as

$$-\Delta u + V(\epsilon x)u = f(u). \tag{1.10}$$

We will deal with this version of (1.2) exclusively. Extend f and F to the negative reals by defining $f(-q) = f(q)$. Let $E = W^{1,2}(\mathbf{R}^n)$ and define $I^\epsilon \in C^1(E, \mathbf{R})$ by $I^\epsilon(u) = \int_{\mathbf{R}^n} \frac{1}{2}(\|\nabla u\|^2 + V(\epsilon x)(u)^2) - F(u)\, dt$. We will search for positive critical points of I^ϵ. By elliptic regularity theory, such points are classical (C^2) solutions of (1.10). If V is a constant (Case I), it is well known that such a solution exists. If Case II holds, then work of [FdP2] gives the result. For Case III, we employ an original argument based on the idea of the $n = 1$ proof described above. We construct a sequence (u_m) with similar properties, and show it is "confined" to the "tube" O/ϵ. Then we find a nozero critical point v of I_ϵ with the property that, along a subsequence, u_m is close to a translate of v.

Organization of Paper

In Section 2, we prove Theorem 1.8. Only basic real analysis techniques are used. Section 3 contains some technical results, primarily about sequences of the form (u_m), where $I^{\epsilon_m \prime}(u_m) \to 0$ and $\epsilon_m \to 0$. Section 4 contains the proof of Theorem 1.7.

2. Almost Periodic Functions on R^2

In this section we prove Theorem 1.8. Note that Theorem 1.8 is false for $n \geq 3$; consider the counterexample $V(x_1, x_2, x_3) = 2 + \sin x_1$. Though this theorem is a simple result, the proof is difficult. We will prove the following stronger result:

THEOREM 2.0 *Let $V : \mathbf{R}^2 \to \mathbf{R}$ be almost periodic. Then one of the following four alternatives holds:*
(i) $V \equiv$ constant

(ii) There exists a bounded open set $O \subset \mathbf{R}^2$ with $\inf_{\partial O} V > \inf_O V$

(iii) There exists a bounded open set $O \subset \mathbf{R}^2$ with $\sup_{\partial O} V < \sup_O V$

(iv) There exist $b > \inf_{\mathbf{R}^2} V$, a nonzero vector $\vec{u} \in \mathbf{R}^2$, and a connected set $C \subset \mathbf{R}^2$ and such that $V(x) > b$ for all $x \in C$, $C \cdot \vec{u} \equiv \{x \cdot \vec{u} \mid x \in C\}$ is bounded, and if $\vec{v} \neq 0$ with $\vec{v} \cdot \vec{u} = 0$, then $C \cdot \vec{v} = \mathbf{R}$.

This gives Theorem 1.8 for the following reason: in Case(2.0)ii, (1.7)(II) is satisfied; pick $z \in O$ with $V(z) = \min_O V$ and take $B_0 = \emptyset$, $B = \{z\}$, and $\Lambda = O$, choosing O to be smooth enough. In Case(2.0)(iii), (1.7)(II) and (III) *both* hold. For Case (1.7)(iv), let $b_2 \in (\inf_{\mathbf{R}^2} V, b)$. Let R be large enough so that every ball of radius R in $\mathbf{R}^2$ contains a point z with $V(z) < b_2$. This is possible because V is a.p. Let $a \in \mathbf{R}^2$ be a $b_2 - b$-almost period of V with $a \cdot \vec{u}$ large enough so that $\mathbf{R}^2 \setminus (C \cup (C + a))$ contains a component which is bounded in the $\vec{u}$ direction, and which contains a ball of radius R. Let z be a point in that ball with $V(z) < b_2$. Let O be the component of $V^{<b} \equiv \{x \mid V(x) < b\}$ containing z. Since $V > b_2$ on $C \cup (C + a)$, O is bounded in the $\vec{u}$ direction.

We will need to use another definition of almost periodic which is equivalent ([Be]) to that given in the Introduction. "Almost periodic" is easily generalized to functions from $\mathbf{R}^n$ into an arbitrary Banach space (the domain may also be generalized, but we need not consider this). For $V : \mathbf{R}^n \to X$ and $x \in \mathbf{R}^n$, define $\tau_x V : \mathbf{R}^n \to X$ by $\tau_x V(y) = V(y - x)$. That is, $\tau_x V$ is V translated by x. Let $C(\mathbf{R}^n, X)$ denote the Banach space of bounded, continuous functions from $\mathbf{R}^n$ to X, with the uniform norm. A continuous function $V : \mathbf{R}^n \to X$ is almost periodic if and only if the set of translates $\{\tau_x V \mid x \in \mathbf{R}^n\}$ is precompact in $C(\mathbf{R}^n, X)$; that is, if any sequence $(x_m) \subset \mathbf{R}^n$ has a subsequence (x_m) with $\tau_{x_m} V$ convergent in $C(\mathbf{R}^n, X)$.

This alternate definition will help us prove the following lemma, which proves the less-than-obvious fact that if $V : \mathbf{R}^n \to \mathbf{R}$ is almost periodic, then it is almost periodic "in any direction:"

LEMMA 2.1 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be almost periodic, $\vec{u} \in \mathbf{R}^n$, and $\epsilon > 0$. Then there exists a relatively dense set $A \subset \mathbf{R}$ such that for all $a \in A$, $a\vec{u}$ is an ϵ-period of V.*

Proof: without loss of generality let $\vec{u} = (1, 0, 0, \ldots, 0)$. Define $\mathcal{V} : \mathbf{R} \to C(\mathbf{R}^{n-1}, \mathbf{R})$ by $\mathcal{V}(x)(y) = V(x, y)$. We will show $\mathcal{V}$ is almost periodic. Let $(x_m) \subset \mathbf{R}$. Since V is a.p., $\{\tau_{(x_m,0)} V \mid m \geq 1\}$ is precompact in $C(\mathbf{R}^n, \mathbf{R})$. Take a subsequence of (x_m) and $\bar{V} \in C(\mathbf{R}^n, \mathbf{R})$ with $\tau_{(x_m,0)} V \to \bar{V}$ in $C(\mathbf{R}^n, \mathbf{R})$. Define $\bar{\mathcal{V}} \in C(\mathbf{R}, C(\mathbf{R}^{n-1}, \mathbf{R}))$ by $\bar{\mathcal{V}}(x)(y) = \bar{V}(x, y)$. It is easy to see that $\tau_{x_m} \mathcal{V} \to \bar{\mathcal{V}}$ in $C(\mathbf{R}, C(\mathbf{R}^{n-1}, \mathbf{R}))$. Therefore, $\mathcal{V}$ is almost periodic, and there exists a relatively dense set $A \subset \mathbf{R}$ such that for all $a \in A$, $\|\tau_a \mathcal{V} - \mathcal{V}\|_{C(\mathbf{R}, C(\mathbf{R}^{n-1}, \mathbf{R}))} < \epsilon$. From this it is clear that for all $(x, y) \in \mathbf{R}^n$, $|V(x - a, y) - V(x, y)| < \epsilon$. Therefore $(a, 0)$ is an ϵ-period of V.

We need several lemmas to complete the proof of Theorem 2.0. First,

LEMMA 2.2 *Let $V : \mathbf{R}^2 \to \mathbf{R}$ be almost periodic and not satisfy any of Cases (2.0)(i)-(iii). Suppose that there exist $\vec{u} \in \mathbf{R}^2 \setminus \{(0, 0)\}$, and an unbounded, connected set $C \subset \mathbf{R}^2$ such that $C \cdot \vec{u}$ is bounded, and $\inf_C V > \inf_{\mathbf{R}^2} V$ or $\sup_C V < \sup_{\mathbf{R}^2} V$. Then (2.0)(iv) holds.*

Proof: Let $\vec{v} \in \mathbf{R}^2 \setminus \{0\}$ with $\vec{v} \perp \vec{u}$. First let us take the case $\inf_{\mathbf{R}^2} C > b > b_1 > b_2 > \inf_{\mathbf{R}^2} V$. Since C is unbounded but $C \cdot \vec{u}$ is bounded, $C \cdot \vec{v}$ is unbounded (and connected). $C \cdot \vec{v}$ contains an interval of the form $(-\infty, \lambda)$ or (λ, ∞). Assume without loss of generality that the former occurs. Let $(a_m) \subset \mathbf{R}$ be a sequence with $a_m \to \infty$ and $a_m \vec{v}$ a $(b - b_1)$-almost period of V for each m. This is possible by Lemma 2.1.

Let $R>0$ be large enough so that $C\cdot\vec{u}\subset(-R,R)$. For large m, $C+a_m\vec{v}\equiv\{x+a_m\vec{v}\mid x\in C\}$ intersects the segment $\overline{-R\vec{u}\;R\vec{u}}$. Also, $V>b_1$ on $C+a_m\vec{v}$. Let $\rho_m\in(-R,R)$ with $\rho_m\vec{v}\in\overline{-R\vec{u}\;R\vec{u}}\cap(C+a_m\vec{v})$. Take a subsequence along which ρ_m converges to $\bar\rho\in[-R,R]$. $V(\bar\rho\vec{u})\geq b_1$.

Let $r>0$ be small enough so that $V>b_2$ on $B_r(\bar\rho\vec{u})$. Take a "tail" of the sequence (a_m), so that $\rho_m\vec{v}\in B_r(\bar\rho\vec{u})$ for all m. Define

$$A=B_r(\bar\rho\vec{u})\cup\bigcup_{m=1}^{\infty}(C+a_m\vec{v}). \tag{2.3}$$

Now $A\cdot\vec{u}$ is bounded, because $C\cdot\vec{u}$ is bounded and $\vec{v}\perp\vec{u}$. $A\cdot\vec{v}=\mathbf{R}$, because $C\cdot\vec{v}$ is unbounded in the negative direction and $a_m\to\infty$. $V>b_2$ on A, since $V>b_2$ on $B_r(\bar\rho\vec{v})$ and $V>b_1>b_2$ on $C+a_m\vec{v}$ for each m. A is connected because $C+a_m\vec{v}$ intersects $B_r(\bar\rho\vec{v})$ for all m. (2.0)(iv) follows.

Now for the case $\sup_C V<\sup_{\mathbf{R}^2}V$. By the above argument, we can construct $C_2\subset\mathbf{R}^2$ with C_2 connected, $\sup_{C_2}V<\sup_{\mathbf{R}^2}V$, $C_2\cdot\vec{u}$ bounded, and $C_2\cdot\vec{v}=\mathbf{R}$. Also, there exists a translate of C_2, called C_3, with the property that $\mathbf{R}^2\setminus(C_2\cup C_3)$ has a component which is bounded in the $\vec{u}$ direction and which contains a point z with $V(z)>b>\sup_{C_2\cup C_3}V$. Let U denote the component of $\{V>b\}$ containing z. U is sandwiched between C_2 and C_3, so $U\cdot\vec{u}$ is bounded. Since (2.0)(iii) does not hold, U is unbounded. Applying the first part of Lemma 2.2, just proven, (2.0)(iv) follows.

Next, define $V^{<b}=\{x\in\mathbf{R}^2\mid V(x)<b\}$ and $V_{>b}=\{x\in\mathbf{R}^2\mid V(x)>b\}$. We say that $x,y\in V^{<b}$ are "b-connected" if they are in the same component of $V^{<b}$. Then,

LEMMA 2.4 *Let $V:\mathbf{R}^2\to\mathbf{R}$ be almost periodic, and not satisfy any of 2.0(i)-(iv). Let $b\in(\inf_{\mathbf{R}^2}V,\sup_{\mathbf{R}^2}V)$. Then there exists $R=R(b,V)$ with the following property: if $x,y\in V^{<b}$ and are b-connected, then for all z in the segment $\overline{xy}$, there exists $u\in B_R(z)\cap V^{<b}$ that is b-connected with x and y.*

Proof: let $b\in(\inf_{\mathbf{R}^2}V,\sup_{\mathbf{R}^2}V)$ and $b_2\in(b,\sup_{\mathbf{R}^2}V)$. Pick $R>0$ big enough so that for all $a\in\mathbf{R}^2$, $B_{R/2}(a)$ contains a (b_2-b)-almost period of V.

Let $x,y\in V^{<b}$ be b-connected, and $z\in\overline{xy}$. Suppose that $B_R(z)$ contains no points in $V^{<b}$ that are b-connected with x and y. Let $\bar\gamma:[0,1]\to V^{<b}\setminus B_R(z)$ be a continuous path with $\bar\gamma(0)=x$ and $\bar\gamma(1)=y$.

Let $\vec{v}=(y-x)/|y-x|$ and $|\vec{u}|=1$ with $\vec{u}\perp\vec{v}$. By taking a subset of $\bar\gamma([0,1])$, we can find a path $\gamma:[0,1]\to\mathbf{R}^2\setminus B_R(z)$ and points x',y' with

(2.5)(i) $$\gamma(0)=x',\ \gamma(1)=y'$$

(ii) $$x' \text{ and } y' \text{ are on the line } \overleftrightarrow{xy}$$

(iii) $$(\gamma(t)-z)\cdot\vec{u}\neq 0 \text{ for all } t\in(0,1),$$

(iv) $$(x'-z)\cdot\vec{v}\leq -R,\quad (y'-z)\cdot\vec{v}\geq R.$$

By (2.5)(iii) we may assume without loss of generality that $(\gamma(t)-z)\cdot\vec{u}>0$ for all $t\in(0,1)$.

Let U be any component of $V_{>b_2}$. By Lemma 2.2 and the assumption that (2.0)(iv) is false, U is unbounded in every direction, in particular the direction $\vec{u}$. Let $D=\max\{|\gamma(t_1)-\gamma(t_2)|\mid t_1,t_2\in[0,1]\}$, the diameter of $\gamma([0,1])$. Let $\bar{g}:[0,1]\to U$ with $(\bar{g}(1)-\bar{g}(0))\cdot\vec{u}>D$. By taking a subset of $\bar{g}([0,1])$, we obtain a path $g:[0,1]\to U\subset V_{>b_2}$ and $w_0, w_1\in U$ with

$$(2.6)(i) \qquad w_0 = g(0),\ w_1 = g(1)$$

$$(ii) \qquad (w_1 - w_0)\cdot\vec{u} > D$$

$$(iii) \qquad (g(t) - w_0)\cdot\vec{u} > 0 \text{ for all } t\in(0,1].$$

Let P denote the closed half-plane $\{\xi \mid (\xi - z)\cdot\vec{u}\geq 0\}$. $P\cap B_R(z)$ contains a ball of radius $R/2$. By the choice of R, there exists $y\in P\cap B_R(z)$ with $y-w_0$ a (b_2-b)-almost period of V. Define $g_2:[0,1]\to\mathbf{R}^2$ by $g_2(t)=g(t)+(y-w_0)$. Then $V>b_2-(b_2-b)=b$ on $g_2([0,1])$. By (2.6)(iii), $g_2([0,1])$ lies in the half-plane P. $\gamma([0,1])$ separates P into two components. One is a bounded component that contains $B_R(z)\cap P$. $g_2(0)=y\in B_R(z)\cap P$. By (2.6)(ii), $g_2(1)$ belongs to the unbounded component of $P\setminus\gamma([0,1])$. Therefore the path g_2 must cross the path γ. This is impossible, since $V<b$ on $\gamma([0,1])$ and $V>b$ on $g_2([0,1])$. Lemma 2.4 is proven.

Lemma 2.4 helps prove the following lemma:

Lemma 2.7 *Let $V:\mathbf{R}^2\to\mathbf{R}$ be almost periodic, and not satisfy any of (2.0)(i)-(iv). Then there exist $b\in(\inf_{\mathbf{R}^2}V,\sup_{\mathbf{R}^2}V)$ and an unbounded sequence of colinear points in $V^{<b}$ that are b-connected.*

Proof: let $b\in(\inf_{\mathbf{R}^2}V,\sup_{\mathbf{R}^2}V)$. Assume, by translating V if necessary, that $V(0,0)<b$. Since (2.0)(ii) is false, there exists an unbounded sequence of points $(x_m)\subset V^{<b}$ that are b-connected with $(0,0)$. Taking a subsequence, and rotating V if necessary, we may assume that $x_m/|x_m|\to(1,0)$ as $m\to\infty$.

Let $R=R(V,b)$ be as in Lemma 2.4. Fix $l\in\mathbf{N}$. For large m, $|x_m|>2lR$ and there exists $z_m\in B_R(2lR\frac{x_m}{|x_m|})$ that is b-connected with $(0,0)$. For large enough m, $z_m\in B_R(2lR\frac{x_m}{|x_m|})\subset B_{2R}(2lR,0)$. In this manner, construct a sequence of distinct points (z_m) that are b-connected to $(0,0)$, with $z_m\subset B_{2R}(2l_mR,0)$ and $l_m\to\infty$.

Let $z_m=(z_m^{(1)},z_m^{(2)})$. $z_m^{(2)}\in(-2R,2R)$, so we can extract a subsequence $(w_m)=(w_m^{(1)},w_m^{(2)})\subset(z_m)$ with (w_m) unbounded, each w_m b-connected to $(0,0)$, and $w_m^{(2)}\to\rho\in[-2R,2R]$ as $m\to\infty$. Let $b_2\in(b,\sup_{\mathbf{R}^2}V)$. V is uniformly continuous (see [Be]), so let $r>0$ be small enough so that $V<b_2$ on $B_r(w_m)$ for all m. For large m, we may select $y_m\equiv(y_m^{(1)},y_m^{(2)})\in B_r(w_m)$ with $y_m^{(2)}=\rho$, $V(y_m)<b_2$, and y_m b_2-connected to w_m, hence to $(0,0)$. The lemma is proven (with b_2 in place of b).

We state the following result and postpone the proof until later:

Lemma 2.8 *Let $0<r<l$ and $M\geq 4l^2/r$, and let $\gamma\in C([0,1],\mathbf{R}^2)$ be a one-to-one path with $\gamma(0)=(0,0)$ and $\gamma(1)=(M,0)$. Then there exist $t_1,t_2\in[0,1]$ with*

$$|\gamma(t_1)-(\gamma(t_2)+(l,0))|\leq r.$$

Now we complete the proof of Theorem 2.0. Suppose none of Cases (2.0)(i)-(iv) hold. By Lemma 2.7, there exist $b\in(\inf_{\mathbf{R}^2}V,\sup_{\mathbf{R}^2}V)$ and an unbounded sequence (x_m) of colinear points in $\mathbf{R}^2$ with $V(x_m)<b$

for all m and all the x_m's b-connected to each other. Without loss of generality, by rotating and translating V, taking a subsequence, and abusing notation, we may assume that $x_m = (x_m, 0)$, $x_1 = (0,0)$, and $x_m \to \infty$.

Let $b_2 \in (b, \sup_{\mathbf{R}^2} V)$ and let $r > 0$ be small enough so that for all $x, y \in \mathbf{R}^2$,

$$|x - y| \le r \Rightarrow |V(x) - V(y)| < (b_2 - b)/2. \tag{2.9}$$

Let $L > 0$ be large enough so that for all $t \in \mathbf{R}$, there exists $t_0 \in (t, t + L/2)$ with $(t_0, 0)$ a $(b_2 - b)/2$-almost period of V. This is possible by Lemma 2.1.

Fix an m with $x_m > 4L^2/r$. Let C be the image of a one-to-one path connecting $(0,0)$ and $(x_m, 0)$ in $V^{<b}$. By the choice of L above, there exists a sequence $(a_m) \subset \mathbf{R}$ with $a_m \to \infty$, $|a_{i+1} - a_i| < L$ for all $i \ge 1$, and $(a_i, 0)$ a $(b_2 - b)/2$-almost period of V for all i.

Define

$$A = \bigcup_{i=1}^{\infty} N_r(C + (a_i, 0)). \tag{2.10}$$

A is unbounded, since (a_i) is unbounded. A is bounded in the x_2-direction, since C is bounded. If $z \in A$, then z has the form $z = x + (a_i, 0) + y$ for some $x \in C$ and $y \in B_r((0,0))$, with $(a_i, 0)$ a $(b_2 - b)/2$-almost period of V. By (2.9), $V(z) \le V(x) + (b_2 - b)/2 + (b_2 - b)/2 < b + (b_2 - b) = b_2$, so $\sup_A V \le b_2 < \sup_{\mathbf{R}^2} V$. Finally, A is connected: for each $i \ge 1$, $|a_{i+1} - a_i| < L$. C connects the points $(0,0)$ and $(M, 0) \equiv (x_m, 0)$, with $M > 4L^2/r$. By Lemma 2.8, C and $C + (a_{i+1} - a_i, 0)$ are within distance r of each other. Thus $C + (a_i, 0)$ and $C + (a_{i+1}, 0)$ are within distance r of each other for each $i \ge 1$. Thus $N_r(C + (a_i, 0))$ and $N_r(C + (a_{i+1}, 0))$ intersect, and A is connected. By Lemma 2.2, (2.0)(iv) holds. This contradicts our assumption that (2.0)(i)-(iv) do not hold, proving Theorem 2.0.

Proof of Lemma 2.8:

If suffices to prove the result for $l = 1$, because if $l' \ne 1$ and $0 < r' < l'$, we may apply the lemma with $r = r'/l'$, $l = 1$, and $M = 4/r'$, then rescale by l' to obtain the desired result.

From now on we assume that the conclusion of Lemma 2.8 does not hold, that is,

$$C \text{ and } C + (1,0) \text{ are separated by distance at least } r. \tag{2.11}$$

We will obtain a contradiction. Assume without loss of generality that

$$\gamma([0,1]) \cap (((-\infty, 0) \cup (M, \infty)) \times \{0\}) = \emptyset. \tag{2.12}$$

If this were not true, then we could take $x_- = \min\{x \mid (x,0) \in \gamma([0,1])\} \le 0$ and $x_+ = \max\{x \mid (x,0) \in \gamma([0,1])\} \ge 1$ and apply the conclusion of Lemma 2.8 to the "sub-arc" of γ connecting $(x_-, 0)$ and $(x^+, 0)$ (shifted $|x_-|$ units to the right if necessary).

Define projection operators $\pi_1, \pi_2 : \mathbf{R}^2 \to \mathbf{R}$ by $\pi_1(x, y) = x$, $\pi_2(x, y) = y$. Define $y_{\max} = \max\{\pi_2(\gamma(s)) \mid x \in [0,1]\}$ and $y_{\min} = \min\{\pi_2(\gamma(s)) \mid x \in [0,1]\}$. Now $y_{\max} > 0 > y_{\min}$; otherwise, $\gamma([0,1])$ and $\gamma([0,1]) + (1,0)$ would intersect, contradicting (2.11).

Let t_0 and t_1 satisfy $\pi_2(\gamma(t_0)) = y_{\max}$, $\pi_2(\gamma(t_1)) = y_{\min}$, and $\pi_2(\gamma(t)) \in (y_{\min}, y_{\max})$ for all t strictly between t_0 and t_1. Assume without loss of generality that $t_0 < t_1$.

$\gamma([t_0, t_1])$ separates the strip $\mathbf{R} \times [y_{\min}, y_{\max}]$ into two components. Let U_L denote the component of $(\mathbf{R} \times [y_{\min}, y_{\max}]) \setminus \gamma([t_0, t_1])$ that is unbounded in the negative x_1-direction, and U_R, the other component of $(\mathbf{R} \times [y_{\min}, y_{\max}]) \setminus \gamma([t_0, t_1])$ (which is unbounded in the positive x_1-direction).

We claim that for some $s \in (t_0, t_1)$, $\gamma(s) \in (0, 1) \times \{0\}$. Proof: otherwise, (2.12) implies that $(1, 0)$ lies in U_L. $(1, 0)$ and $\gamma(t_0) + (1, 0)$ both are on the path $\gamma + (1, 0)$. $\gamma([0, 1]) + (1, 0)$ is disjoint with $\gamma([t_0, t_1])$. Thus $(1, 0)$ and $\gamma(t_0)$ are in the same component of $(\mathbf{R} \times [y_{\min}, y_{\max}]) \setminus \gamma([t_0, t_1])$. Clearly, by the definition of U_R, $\gamma(t_1) + (1, 0) \in U_R$. This is a contradiction, and the claim is proven. By similar reasoning, $\gamma([t_0, t_1]) \cap ((M - 1, M) \times \{0\}) \neq \emptyset$.

Next, we claim that for all $x \in (1, M - 2)$,

$$((x, x + 1) \times \{0\}) \cap \gamma([t_0, t_1]) \neq \emptyset. \tag{2.13}$$

To prove, suppose that for some $x \in (1, M - 2)$, (2.13) does not hold. Let $a \leq x$ and $b \geq x + 1$ with $(a, 0), (b, 0) \in \gamma([t_0, t_1])$ but $(y, 0) \notin \gamma([t_0, t_1])$ for all $y \in (a, b)$. This is possible because $\gamma([t_0, t_1])$ intersects both $(0, 1) \times \{0\}$ and $(M - 1, M) \times \{0\}$. Let t_a and t_b satisfy $\gamma(t_a) = (a, 0)$, $\gamma(t_b) = (b, 0)$, and $\pi_2(\gamma(t)) \neq 0$ for all t strictly between t_a and t_b. $\pi_2(\gamma(t))$ has a single, nonzero sign for all t strictly between t_a and t_b. Since $b - a \geq 1$, $\gamma([t_a, t_b])$ must intersect $\gamma([t_a, t_b]) + (1, 0)$, violating (2.11). Claim (2.13) is proven.

For $m = 0, 1, 2, \ldots$, define $A_m \subset [0, M]$ by

$$A_m = \{x \in [0, M] \mid (x, 0) \in \overline{U_L} + (m, 0)\}. \tag{2.14}$$

By assumption (2.11), and the definition of t_0 and t_1, it is apparent that

$$N_r(A_m) \cap [0, M] \subset A_{m+1} \tag{2.15}$$

for $m = 0, 1, 2, \ldots$. Let $m^* = \lfloor 2/r \rfloor + 1$, where '$\lfloor x \rfloor$' denotes the greatest integer less than or equal to x. Since $m^* \geq 2/r$, (2.13) and (2.15) imply that $A_{m^*} = [0, M]$. Now $m^* + 1 = \lfloor 2/r \rfloor + 2 \leq 2/r + 2 < 4/r \leq M$, so $m^* + 1 \in A_{m^*}$. By definition of A_{m^*+1}, $(m^* + 1, 0) \in \overline{U_L} + (m^*, 0)$, so $(1, 0) \in \overline{U_L}$. We have seen that this is false. Therefore, assumption (2.11) is false, proving Lemma 2.8.

3. Technical Results

To prove Theorem 1.7 we seek critical points of I^ϵ, as defined in the Introduction. This will require a detailed study of Palais-Smale sequences of I^ϵ, that is, sequences (u_m) with $I^\epsilon(u_m)$ convergent and $I^{\epsilon\prime}(u_m) \to 0$ as $m \to \infty$. It is well-known that I^ϵ badly fails the "Palais-Smale condition," that is, a Palais-Smale condition need not be precompact. We will be able to examine the structure of such sequences, however, employing concentration-compactness ideas like those developed in [Lio]. We need to obtain estimates on such sequences that are independent of ϵ. Therefore we will examine sequences of the form $(u_m; I^{\epsilon_m})$, where $I^{\epsilon_m}(u_m)$ converges, $I^{\epsilon_m\prime}(u_m) \to 0$, and $\epsilon_m \to 0$. We will obtain results similar to those in [CR2], which involved Palais-Smale sequences of a functional containing a periodic coefficient function. Our proofs will be very similar. Because the functional I^{ϵ_m} varies, however, extra care is required, so we give the proofs in some detail.

If $V : \mathbf{R}^n \to \mathbf{R}^+$ is measurable, bounded, and bounded away from zero, define $(\cdot,\cdot)_V : E \times E \to \mathbf{R}$ by $(u,w)_V = \int_{\mathbf{R}^n} \nabla u \cdot \nabla w + V(x)uw\,dx$. $(\cdot,\cdot)_V$ is equivalent to the usual inner product on $W^{1,2}(\mathbf{R}^n)$. Also define $\|u\|_V = \sqrt{(u,u)_V^2}$, and $I[V] : E \to \mathbf{R}$ by $I[V](u) = \frac{1}{2}\|u\|_V^2 - \int_{\mathbf{R}^n} F(u)$. Then,

PROPOSITION 3.0 *Let f satisfy $(f_1)-(f_4)$. Let (V_m) be positive, uniformly continuous functions on $\mathbf{R}^n$ that are uniformly bounded and uniformly bounded away from zero, and define $I_m = I[V_m]$. Let (u_m) be a sequence in E with $(I_m(u_m))$ bounded and $I_m'(u_m) \to 0$ as $m \to \infty$. Then there exists a subsequence (also denoted (u_m)) such that either (a) $u_m \to 0$, or (b), there exists $u_\infty \in E \setminus \{0\}$, a sequence $(x_m) \subset \mathbf{R}^n$, and $V_\infty : \mathbf{R}^n \to \mathbf{R}$ satisfying*

$$(i)\ \tau_{-x_m} V_m \to V_\infty \text{ locally uniformly as } m \to \infty$$

and defining $I_\infty = I[V_\infty]$,

$$(ii)\ I_\infty'(u_\infty) = 0$$

$$(iii)\ I_m'(u_m - \tau_{x_m} u_\infty) \to 0$$

$$(iv)\ I_m(u_m - \tau_{x_m} u_\infty) \to \lim_{m\to\infty} I_m(u_m) - I_\infty(u_\infty)$$

Proof: Suppose (a) does not hold, and $(\|u_m\|)$ is bounded away from zero. By arguments from [CR1-2], (u_m) is bounded. Also by [CR2], there exists $\rho > 0$, a sequence $(x_m) \subset \mathbf{R}^2$, and a subsequence of (u_m) (also denoted (u_m)), such that $\|u_m\|_{L^2(B_1(x_m))} \geq \rho$ for all m. By translating the u_m's and the V_m's, we may assume that $x_m \equiv 0$. Assume that $n \geq 3$ (the $n = 1, 2$ arguments are very similar). Since (u_m) is bounded, there exists $u_\infty \in E$ such that, along a subsequence, $u_m \rightharpoonup u_\infty$ weakly in $L^2(\mathbf{R}^2)$ and in $W^{1,2}(\mathbf{R}^n)$, and $u_m \to u_\infty$ in L^r_{loc} for every $r \in [1, 2n/(n-2))$. Therefore $\|u_\infty\|_{L^2(B_1(0))} \geq \rho$, and $u_\infty \neq 0$. Since the V_m's are uniformly continuous and uniformly bounded, there exist V_∞ and a subsequence of (V_m) with $V_m \to V_\infty$ locally uniformly in $\mathbf{R}^2$.

To prove (3.0)(iv), take a subsequence of (u_m) along which $I_m(u_m)$ converges and let $b = \lim_{m\to\infty} I_m(u_m)$. Modifying Lemma 1.21 of [CR1] slightly shows $b > 0$. By the dominated convergence theorem, $I_m(u_\infty) - I_m(u_m) \to 0$ and $I_m(u_\infty) \to b$ as $m \to \infty$. So to prove (3.0)(iv), it suffices to show $b - (I_m(u_\infty) + I_m(u_m - u_\infty)) \to 0$. Defining $(u,w)_m \equiv (u,w)_{V_m}$ and $\|u\|_m \equiv \sqrt{(u,u)_m}$,

$$\begin{aligned} b-(I_m(u_\infty) + I_m(u_m - u_\infty)) &= I_m(u_m) - (I_m(u_\infty) + I_m(u_m - u_\infty)) = \\ &= \frac{1}{2}(\|u_m\|_m^2 - (\|u_\infty\|_m^2 + \|u_m - u_\infty\|_m^2)) + \int_{\mathbf{R}^n} F(u_\infty) + F(u_m - u_\infty) - F(u_m) = \\ &= (u_\infty, u_m - u_\infty)_m + \int_{\mathbf{R}^n} F(u_\infty) + F(u_m - u_\infty) - F(u_m). \end{aligned} \tag{3.1}$$

The inner product goes to zero as $m \to \infty$ because $u_m - u_\infty \rightharpoonup 0$ in E. Let $R > 0$. By (f_3), Sobolev estimates, and estimates in [R3],

$$\begin{aligned} \Big|\int_{\mathbf{R}^n} F(u_\infty) + F(u_m - u_\infty) - F(u_m)\Big| &= \Big|\int_{B_R(0)} F(u_\infty) - F(u_m) + \int_{B_R(0)} F(u_m - u_\infty) + \\ &+ \int_{B_R(0)^{\mathbf{C}}} F(u_m - u_\infty) - F(u_m) + \int_{B_R^{\mathbf{C}}(0)} F(u_\infty)\Big| \leq \\ &\leq o(m) + \left|\int_{B_R^{\mathbf{C}}(0)} F(u_m - u_\infty) - F(u_m) + \int_{B_R(0)^{\mathbf{C}}} F(u_\infty)\right| \leq \\ &\leq o(m) + C\|u_\infty\|_{W^{1,2}(B_R(0)^{\mathbf{C}})} + \int_{B_R(0)^{\mathbf{C}}} F(u_\infty) \end{aligned} \tag{3.2}$$

with $o(m) \to 0$ as $m \to \infty$, for some C independent of R and m, using the fact that (u_m) is bounded. Letting R be large, we can make $\limsup_{m\to\infty} |\int_{\mathbf{R}^n} F(u_\infty) + F(u_m - u_\infty) - F(u_m)|$ as small as we like, proving (3.0)(iv).

To help prove both (3.0)(ii) and (iii), we will show

$$I'_m(u_\infty) - I'_\infty(u_\infty) \to 0 \tag{3.3}$$

as $m \to \infty$. Let $\epsilon_1 > 0$. Let $R > 0$ be large enough so that $\|u_\infty\|_{W^{1,2}(B_R(0)^{\mathbf{C}})} < \epsilon_1$. Let $z \in E$. Then

$$\begin{aligned} |(I'_m(u_\infty) - I'_\infty(u_\infty))z| &= |\int_{\mathbf{R}^n} (V_m - V_\infty)u_\infty z| \le \\ &\le |\int_{B_R(0)} (V_m - V_\infty)u_\infty z| + |\int_{B_R(0)^{\mathbf{C}}} (V_m - V_\infty)u_\infty z| \le \\ &\le \int_{B_R(0)} |V_m - V_\infty||u_\infty||z| + V^+ \int_{B_R(0)^{\mathbf{C}}} |u_\infty||z| \le \\ &\le (\sup_{x\in B_R(0)} |V_m(x) - V_\infty(x)|)\|u_\infty\|_{L^2(B_R(0))}\|z\|_{L^2(B_R(0))} + \\ &\qquad + V^+\|u_\infty\|_{L^2(B_R(0)^{\mathbf{C}})}\|z\|_{L^2(B_R(0)^{\mathbf{C}})} \le \\ &\le (\sup_{x\in B_R(0)} |V_m(x) - V_\infty(x)| \cdot \|u_\infty\| + V^+\epsilon_1)\|z\|. \end{aligned} \tag{3.4}$$

Since $V_m \to V_\infty$ locally uniformly, $\limsup_{m\to\infty} \|I'_m(u_\infty) - I'_\infty(u_\infty)\| \le V^+\epsilon_1$. ϵ_1 is arbitrary, so (3.3) follows.

Now to prove (3.0)(ii), note that

$$\begin{aligned} \|I'_\infty(u_\infty)\| &= \|I'_m(u_m) + (I'_m(u_\infty) - I'_m(u_m)) + (I'_\infty(u_\infty) - I'_m(u_\infty))\| \le \\ &\le \|I'_m(u_\infty) - I'_m(u_m)\| + o(m) \end{aligned} \tag{3.5}$$

by (3.4) and the fact that $I'_m(u_m) \to 0$. Let $z \in E$. We must show $(I'_m(u_\infty) - I'_m(u_m))z \to 0$:

$$\begin{aligned} |(I'_m(u_\infty) - I'_m(u_m))z| &= |(u_\infty - u_m, z)_m + \int_{\mathbf{R}^n} (f(u_m) - f(u_\infty))z| \le \\ &\le |(u_\infty - u_m, z)_m| + \int_{\mathbf{R}^n} |f(u_m) - f(u_\infty)||z|. \end{aligned}$$

Since $u_\infty \rightharpoonup u_m$ weakly in E, $(u_\infty - u_m, z)_m \to 0$. We must show $\int_{\mathbf{R}^n} |f(u_m) - f(u_\infty)||z| \to 0$. Let $R > 0$. Then

$$\int_{\mathbf{R}^n} |f(u_m) - f(u_\infty)||z| = \int_{B_R(0)} |f(u_m) - f(u_\infty)||z| + \int_{B_R(0)^{\mathbf{C}}} |f(u_m) - f(u_\infty)||z|.$$

Since $u_m \to u_\infty$ in $L^r_{loc}(\mathbf{R}^2)$ for all $r \in [1, 2n/(n-2))$, (f_3) implies that $\int_{B_R(0)} |f(u_m) - f(u_\infty)||z| \to 0$ as $m \to \infty$. The last integral also approaches 0 as $R \to \infty$, independently of m, by arguments of [R3]. (3.0)(ii) follows.

To prove (3.0)(iii), it suffices, by (3.0)(ii), (3.3) and the fact that $I'_m(u_m) \to 0$, to prove

$$I'_m(u_m) - (I'_m(u_\infty) + I'_m(u_m - u_\infty)) \to 0 \tag{3.6}$$

as $m \to \infty$. Let $\epsilon_1 > 0$ and let R be large enough so that $\|u_\infty\|_{L^2(B_R(0)^{\mathbf{C}})} < \epsilon_1$. Let $z \in E$. Then

$$
\begin{aligned}
[I_m'(u_m) - (I_m'(u_\infty) + I_m'(u_m - u_\infty))]z &= \int_{\mathbf{R}^n} (f(u_\infty) + f(u_m - u_\infty) - f(u_m))z = \\
&= \int_{B_R(0)} (f(u_\infty) - f(u_m))z + \int_{B_R(0)} f(u_m - u_\infty)z + \\
&\quad + \int_{B_R(0)^{\mathbf{C}}} (f(u_m - u_\infty) - f(u_m))z + \int_{B_R(0)^{\mathbf{C}}} f(u_\infty)z = \\
&= o(m)\|z\| + \int_{B_R(0)^{\mathbf{C}}} (f(u_m - u_\infty) - f(u_m))z + \int_{B_R(0)^{\mathbf{C}}} f(u_\infty)z.
\end{aligned}
$$

By (f_3) and arguments of [R3] again, we obtain

$$\|I_m'(u_m) - (I_m'(u_\infty) + I_m'(u_m - u_\infty))\| \le o(m) + C\|u_\infty\|_{W^{1,2}(B_R(0)^{\mathbf{C}})}$$

for some C independent of m and R. Letting $R \to \infty$, (3.0)(iii) is proven.

From Proposition 3.0 comes:

PROPOSITION 3.7 *Let f, (V_m) and I_m be as in Proposition 3.0. Let (u_m) be a sequence in E with $I_m(u_m) \to b > 0$ and $I_m'(u_m) \to 0$ as $m \to \infty$. Then there exist a subsequence of (u_m) (also denoted (u_m)), $k \in \mathbf{N}$, sequences $(x_m^i)_{m\in\mathbf{N}}^{i=1,\dots k}$, $u_\infty^1, \dots, u_\infty^k \in E \setminus \{0\}$, and functions ${V_\infty^i}^{\,i=1,\dots k}$ satisfying*

(i) $|x_m^i - x_m^j| \to \infty$ *for* $i \ne j$ *as* $m \to \infty$

(ii) $\tau_{-x_m^i} V_\infty^i \to V_\infty^i$ *locally uniformly as* $m \to \infty$

and defining $I_\infty^i = I[V_\infty^i]$,

(iii) ${I_\infty^i}'(u_\infty^i) = 0$ *for all* i

(iv) $\|u_m - \sum_{i=1}^k \tau_{x_m^i} u_\infty^i\| \to 0$ *as* $m \to \infty$

(v) $\sum_{i=1}^k I_\infty^i(u_\infty^i) = b$

Proof: By [CR1-2], (u_m) is a bounded sequence. Apply Proposition 3.0 to (u_m), obtaining $(x_m^1) \equiv (x_m)$ and $u_\infty^1 \equiv u_\infty$ as in the conclusion of Proposition 3.0. Let $w_m = u_m - \tau_{x_m^1} u_\infty^1$. Then by Proposition 3.0(iii), $I_m'(w_m) \to 0$, so we may apply Proposition 3.0 again, obtaining (x_m^2) and u_∞^2 satisfying 3.0(i)-(iv). And so on. The process ceases after finitely many steps, because there exists $\underline{b} = \underline{b}(V_-, V^+, f)$ with the property that if $(u_m; V_m)$ is a sequence as in Proposition 3.0, then either (u_m) goes to zero along a subsequence or $\liminf I_m(u_m) \ge \underline{b}$ (to prove this, modify [CR1], Lemma 1.21). Therefore we obtain finite k in (3.7)(v) with $k \le b/\underline{b}$.

Let us apply Proposition 3.7 to the situation at hand. Define V_ϵ by

$$V_\epsilon(x) = V(\epsilon x). \tag{3.8}$$

Then $I^\epsilon = I[V_\epsilon]$.

Proposition 3.9 *Let f satisfy $(f_1)-(f_4)$ and V satisfy $(V_1)-(V_2)$. Let $\epsilon_m \to 0$. Let (u_m) be a sequence in E with $I^{\epsilon_m}(u_m) \to b > 0$ and $I^{\epsilon_m\prime}(u_m) \to 0$ as $m \to \infty$. Then there exist $k \in \mathbf{N}$, $a_i, \ldots, a_k \in [V_-, V^+]$, a subsequence of (u_m) (also denoted (u_m)), $v_1, \ldots, v_k \in E \setminus \{0\}$, and sequences $(x^i_m)^{i=1,\ldots k}_{m\in\mathbf{N}}$ satisfying*

$$(i)\ |x^i_m - x^j_m| \to \infty \text{ as } m \to \infty \text{ for } i \neq j$$

$$(ii)\ V_{\epsilon_m}(x^i_m) \to a_i$$

$$(iii)\ I[a_i]'(v_i) = 0$$

$$(iv)\ \|u_m - \sum_{i=1}^{k} \tau_{x^i_m} v_i\| \to 0$$

$$(v)\ \sum_{i=1}^{k} I[a_i](v_i) = b$$

We will need the following simple lemma in Section 4. It estimates the error incurred when transposing a cutoff function from one side of an inner product to another.

Lemma 3.10 *Let Ω be an open subset of $\mathbf{R}^n$, and $\varphi \in W^{1,\infty}(\Omega)$ with $\|\nabla\varphi\|_{L^\infty(\Omega)} \leq d$. Let V be a measurable function on Ω that is bounded and bounded away from zero. Define $(\cdot,\cdot)_{V;\Omega} : W^{1,2}(\Omega)^2 \to \mathbf{R}$ by $(u,w)_{V;\Omega} = \int_\Omega \nabla u \cdot \nabla v + V(x)uw\,dx$. Then*

$$|(\varphi u, w)_{V;\Omega} - (u, \varphi w)_{V;\Omega}| \leq d\|\nabla u\|_{W^{1,2}(\Omega)}\|\nabla w\|_{W^{1,2}(\Omega)}.$$

for all $u, w \in W^{1,2}(\Omega)$.

Proof:

$$(\varphi u, w)_{V;\Omega} - (u, \varphi w)_{V;\Omega} = \int_\Omega \nabla u \cdot [(\nabla\varphi)w + (\nabla w)\varphi] + u\varphi w\,dx$$
$$- \int_\Omega [(\nabla\varphi)u + (\nabla u)\varphi] \cdot \nabla w + \varphi uw\,dx =$$
$$= \int_\Omega (\nabla\varphi) \cdot [(\nabla u)w - (\nabla w)u]\,dx,$$

so

$$|(\varphi u, w)_{V;\Omega} - (u, \varphi w)_{V;\Omega}| = |\int_\Omega (\nabla\varphi) \cdot [(\nabla u)w - (\nabla w)u]\,dx|$$
$$\leq \|\nabla\varphi\|_{L^\infty(\Omega)} \int_\Omega |\nabla u||w| + |\nabla w||u|\,dx \leq d(\|\nabla u\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}) \leq$$
$$\leq \|\nabla\varphi\|_{L^\infty(\Omega)}\sqrt{\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}}\sqrt{\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}} =$$

(by the Cauchy-Schwartz Inequality)

$$= d\|u\|_{W^{1,2}(\Omega)}\|w\|_{W^{1,2}(\Omega)}.$$

The Gradient Vector Flow

For $V : \mathbf{R}^n \to \mathbf{R}$, define $W_V : E \to E$ to be the gradient of $I[V]$ with respect to the inner product $(\cdot,\cdot)_V$. That is, $(W_V(u), w)_V = I[V]'(u)w$ for all $u, w \in E$. Then define $\eta_I \equiv \eta_{I[V]}$ to be the solution of the initial value problem

$$\frac{d\eta_I}{dt} = -W_V(\eta); \quad \eta_I(0,u) = u. \tag{3.11}$$

It is easy to show that if f satisfies $(f_1) - (f_3)$, then I' is locally Lipschitz continuous, so W_V is locally Lipschitz continous. Thus for every $u \in E$, $\eta_I(t,u)$ exists at least for small $|t|$. We will work with a vector flow of this form in Section 4. We would like to obtain estimates on the norm of $\eta_{I^\epsilon}(t,u)$ that are independent of ϵ, and to find conditions under which $\eta_{I^\epsilon}(t,u)$ is well-defined for all positive t. The latter is not an easy question, since W_{V_ϵ} is not bounded on E, or even on sublevel sets of I^ϵ.

LEMMA 3.12 *Let $B_0 > 0$. There exists $B_2 = B_2(V^+, \mu, B_0)$ with the property that if V satisfies (V_2), $I \equiv I[V]$, and $u \in E$ with $\|u\|_V \le B_0$, then for all $t \ge 0$,*

$$I(\eta_I(t,u)) \ge 0 \Rightarrow \|\eta_I(t,u)\|_V \le B_2. \tag{3.13}$$

Thus for all $u \in E$, either $I(\eta_I(t,u)) < 0$ for some $t > 0$, or $\eta_I(t,u)$ is well-defined and $I(\eta_I(t,u)) \ge 0$ for all $t > 0$.

Proof: To prove the latter claim, suppose $I(\eta_I(t,u)) \ge 0$ for all $t > 0$ in the interval of definition of $\eta_I(\cdot,u)$. If $\eta_I(\cdot,u)$ has an interval of definition of the form (a,b) with $b < \infty$, then there exist (t_m) with $t_m \uparrow b$ and $\|W_V(\eta_I(t_m,u))\| \to \infty$. Since I' is bounded on bounded subsets of E, $\|\eta_I(t_m,u)\| \to \infty$. Since $I(\eta_I(t_m,u)) \ge 0$ for all m, this contradicts (3.13).

To prove (3.13), let $B_0 > 0$. Let $B_1 > B_0$ and be big enough so

$$x \ge B_1 \Rightarrow (\frac{\mu}{2} - 1)x^2 - x > \mu V^+ B_0^2. \tag{3.14}$$

We claim that if $\|w\| \ge B_1$ and $I(w) \le V^+ B_0^2$, then

$$\|I'(w)\|_V \equiv \sup\{I'(w)y \mid \|y\|_V = 1\} > 1; \tag{3.15}$$

to prove, suppose $\|w\|_V = x \ge B_1$, $I(w) \le V^+ B_0^2$, and $\|I'(w)\|_V \le 1$. Then

$$-x = -\|w\|_V \le I'(w)w = \|w\|_V^2 - \int_{\mathbf{R}^n} f(w)w \le \|w\|_V^2 - \mu F(w) = \mu I(w) - (\frac{\mu}{2} - 1)\|w\|_V^2 \le$$
$$\le \mu V^+ B_0^2 - (\frac{\mu}{2} - 1)\|w\|_V^2 = \mu V^+ B_0^2 - (\frac{\mu}{2} - 1)x^2.$$

This contradicts (3.14). (3.15) is proven.

Define

$$B_2 = B_1 + V^+ B_0^2 + 1. \tag{3.16}$$

Suppose $u \in E$ with $\|u\|_V \leq B_0$, and for some $t^* > 0$, $\|\eta_I(t^*, u)\|_V > B_2$ and $I(\eta_I(t^*, u)) \geq 0$. Denote $\eta \equiv \eta(t) \equiv \eta_I(t) \equiv \eta_I(t, u)$. Let $t_1, t_2 \in (0, t^*)$ with $\|\eta(t_1)\|_V = B_1$, $\|\eta(t_2)\|_V = B_2$, and $\|\eta(t)\|_V \in (B_1, B_2)$ for all $t \in (B_1, B_2)$. Then $I(\eta(t)) \in [0, I(u)] \subset [0, V_+ B_0^2]$ for all $t \in (t_1, t_2)$, so

$$V^+ B_0^2 \geq I(\eta(t_1)) - I(\eta(t_2)) = -\int_{t_1}^{t_2} \frac{d}{dt} I(\eta) = \int_{t_1}^{t_2} \|I'(\eta)\|_V^2 \geq$$

$$\geq \int_{t_1}^{t_2} \|I'(\eta)\|_V = \int_{t_1}^{t_2} \left\| \frac{d\eta}{dt} \right\|_V \geq \|\eta(t_2) - \eta(t_1)\|_V \geq B_2 - B_1 = V^+ B_0^2 + 1.$$

This is impossible. (3.13) is proven.

4. Proof of Theorem 1.7

Here we complete the proof of Theorem 1.7, Case III. We follow the plan outlined in the Introduction. Define the *ground energy function* C as follows: for $b > 0$, $C(b)$ is the smallest positive critical value of the functional $I[b]$. By arguments of [R2], $C(b)$ is well defined (even without (f_5)). Because of (f_5), C is continuous and strictly increasing on $(0, \infty)$ (see [WZ]).

Choose b_0, b_1, b_2, b_3, b_4 satisfying

$$\inf_O V = b_0 < b_1 < b_2 < b_3 < b_4 = \inf_{\partial O} V \tag{4.0}$$

and

$$C(b_2) < \frac{3}{2} C(b_0). \tag{4.1}$$

Let $O_{in} \subset O$ with

$$\inf_{O \setminus O_{in}} V > b_3 \tag{4.2}$$

and

$$d(O_{in}, O^{\mathbf{C}}) > 0, \tag{4.3}$$

where $d(A, B) \equiv \sup\{|x - y| \mid x \in A,\ y \in B\}$. (4.2)-(4.3) are possible because V is uniformly continuous.

The most difficult task of this section is to find a Palais-Smale sequence (u_m) for I^ϵ such that $\|u_m - u_{m+1}\| \to 0$ and (u_m) is "confined to O/ϵ". That is, $\|u_m\|_{W^{1,2}((O/\epsilon)^{\mathbf{C}})}$ stays below a certain threshold. Define Θ to be the family of open subsets U of $\mathbf{R}^n$ with the property that $U = int(C)$, the interior of C, where C is a union of closed n-cubes with side a and vertices in the lattice $a\mathbf{Z}^n$ for some $a \geq 1$. Sets in Θ satisfy a uniform cone condition. For $U \subset \mathbf{R}^n$, define the inner product $(u, w)_{V_\epsilon; U} = \int_U \nabla u \cdot \nabla w + V_\epsilon(x) uw \, dx$ on $W^{1,2}(U)$, and the norm $\|u\|_{V_\epsilon; U}^2 = (u, u)_{V_\epsilon; U}$. Then by $(f_1) - (f_3)$, and arguments from [R3], there exists $r_0 > 0$ such that if $U \in \Theta$, $\epsilon > 0$, and $u \geq 0$ with $\|u\|_{V_\epsilon; U} \leq 4r_0$, then

$$\int_U |f(u)||w| \, dx \leq \frac{1}{4} \|w\|_{V_\epsilon; U} r_0. \tag{4.4}$$

Assume also that r_0 is small enough so that if (u_m) is as in Proposition 3.7, then for large enough m there exists x_m with

$$\|u_m\|_{W^{1,2}(B_1(x_m))} > 2r_0. \tag{4.5}$$

This is possible by Proposition 3.7.

Next, let $\epsilon_1 > 0$ and $\delta_0 > 0$ satisfy the following: if $\epsilon \leq \epsilon_1$, $u \in E$, $I(u) < C(b_2)$, and $\|u\|_{V_\epsilon;(O\setminus O_{in})/\epsilon} \geq r_0$, then

$$\|I'(u)\|_{V_\epsilon} > \delta_0. \tag{4.6}$$

To prove this is possible, assume the contrary. Then there exist $\epsilon_m \to 0$ and $(u_m) \subset E$ with $I(u_m) < C(b_2)$, $\|u_m\|_{W^{1,2}((O\setminus O_{in})/\epsilon_m)} \geq \|u_m\|_{V_\epsilon;(O\setminus O_{in})/\epsilon_m}/V^+ \geq r_0/V^+$, and $\|I^{\epsilon_m\prime}(u_m)\|_{V_{\epsilon_m}}$ (defined as in (3.15)) $\to 0$. Since $\min(1, V_-)\|z\|^2_{W^{1,2}(\mathbf{R}^n)} \leq \|z\|^2_{V_\epsilon} \leq V^+\|z\|^2_{W^{1,2}(\mathbf{R}^n)}$ for all $\epsilon > 0$, $\|I^{\epsilon_m\prime}(u_m)\| \to 0$. Apply Proposition 3.9 to obtain $k \geq 1$, sequences (x^i_m), $(a_i)_{i=1,\ldots,k}$, and $(v_i)_{i=1,\ldots,k}$ satisfying 3.9(i)-(v). Since $\|u_m\|_{W^{1,2}((O\setminus O_{in})/\epsilon_m)} \geq r_0/V^+$, it follows for at least one i, the distance $d(x^i_m, (O \setminus O_{in})/\epsilon_m)$ is bounded along a subsequence of m. Since $V > b_3$ on $O \setminus O_{in}$, and $V_{\epsilon_m}(x^i_m) \to a_i$, $a_i \geq b_3$. Since the ground energy function C is monotone, $I[a_i](v_i) \geq C(b_3)$. By Proposition 3.9(v), $\liminf_{m\to\infty} I^{\epsilon_m}(u_m) \geq C(b_3) > C(b_2)$. This contradicts the assumption $I(u_m) < C(b_2)$, proving (4.6).

Let

$$M \in \mathbf{N} \cap (\frac{C(b_2)}{\delta_0 r_0}, \infty). \tag{4.7}$$

Let $\rho = \rho(f, V, M) > 0$ and open sets $O_0, O_1, \cdots, O_M$ satisfy

(4.8)(i) $O_{in} \subset O_0 \subset \overline{O}_0 \subset O_1 \subset \overline{O}_1 \subset \ldots \subset O_M \subset O$.

(ii) $\overline{O}_i$ is a union of closed n-cubes with side ρ and vertices in the lattice $\rho\mathbf{Z}^n$.

(iii) $O_i = int\overline{O}_i$ (the interior of $\overline{O}_i$) for all i.

Assume without loss of generality that

$$0 \in O_{in}, \quad V(0) = b_1. \tag{4.9}$$

By (f_5), $I[b_1]$ has a critical point $\omega \in E$ with $I[B_1](\omega) = C(b_1)$ (see [CR1] for proof) and $\omega(0) = \max\omega$, and there exists $T > 0$ such that $I[b_1](T\omega) = -1$ and $\max_{\theta\in[0,T]} I[b_1](\theta\omega) = C(b_1) = I[b_1](\omega)$. Since $V_\epsilon \to b_1$ locally uniformly as $\epsilon \to \infty$, it is easy to check that for small enough ϵ, $I^\epsilon(T\omega) < -1/2$ and

$$\max_{\theta\in[0,T]} I^\epsilon(\theta\omega) < C(b_2). \tag{4.10}$$

Because of the mountain-pass structure of I^ϵ, it is easy to check that there exists $\theta^\epsilon \in (0, T)$ with

$$\lim_{t\to\infty} \eta_{I^\epsilon}(t, \theta^\epsilon\omega) > 0, \tag{4.11}$$

where η_{I^ϵ} is as in (3.11). Since $\{\eta_{I^\epsilon}(t, \theta^\epsilon\omega) \mid t > 0\}$ is bounded (Lemma 3.12), and $I^{\epsilon\prime\prime}$ is bounded on bounded subsets of E, it is easy to check that

$$\|I^{\epsilon\prime}(\eta_{I^\epsilon}(t, \theta^\epsilon\omega))\|_{V_\epsilon} \to 0 \tag{4.12}$$

as $t \to \infty$. See [STT] for a similar argument.

Recall T from (4.10) and let $C_1 = C_1(T, \omega, V_-, V^+, \mu)$ be large enough so that if $\epsilon > 0$ and $\|u\| \le T\|\omega\|$, then for all $t > 0$ and $\epsilon > 0$,

$$I^\epsilon(\eta_{I^\epsilon}(t,u)) \le 0 \text{ or } \|\eta_{I^\epsilon}(t,u)\|_{V_\epsilon} < C_1. \tag{4.13}$$

This is possible because of Lemma 3.12 and the equivalence of $\|\cdot\|_{V_\epsilon}$ and $\|\cdot\|_{W^{1,2}}$ independently of ϵ. Let $C_2 = C_2(f, V_-, V^+)$ be large enough that if $\epsilon > 0$ and $\|u\|_{V_\epsilon} \le C_1$ then

$$\int_{\mathbf{R}^n} |f(u)||w| \le C_2\|w\|_\epsilon \tag{4.14}$$

for all $w \in E$.

Assume that ϵ is small enough so

$$\|\theta^\epsilon\omega\|_{W^{1,2}(O_{in}/\epsilon)} < \|T\omega\|_{W^{1,2}(O_{in}/\epsilon)} < r_0/V^+. \tag{4.15}$$

Also let

$$\epsilon < \min(\frac{r_0\rho}{8C_2}, \frac{\rho}{2}), \tag{4.16}$$

where ρ is as in (4.8).

For convenience let $\eta \equiv \eta(t) \equiv \eta_{I^\epsilon}(t, \theta^\epsilon\omega)$. We will show that "$\eta$ stays inside O/ϵ," more precisely, that

$$\|\eta(t)\|_{W^{1,2}((O/\epsilon)^{\mathbf{C}})} < r_0 \text{ for all } t > 0. \tag{4.17}$$

To prove this, we assume the contrary. Assume for convenience that

$$V_- \ge 1; \tag{4.18}$$

We can do this without loss of generality: if $V_- < 1$, then (1.10) is equivalent to $-(\epsilon^2/V_-)\Delta u + (V(x)/V_-)u = f(u)/V_-$. Defining $\tilde{\epsilon} = \epsilon/\sqrt{V_-}$, $\tilde{V} = V/V_-$ and $\tilde{f} = f/V_-$, we obtain $-\tilde{\epsilon}^2\Delta u + \tilde{V}u = \tilde{f}(u)$, where $\tilde{V}$ and $\tilde{f}$ satisfy $(V_1) - (V_3)$ and $(f_1) - (f_6)$.

Since $V_- \ge 1$, $\|u\|_{W^{1,2}(U)} \le \|u\|_{V_\epsilon;U}$ for all $u \in E$ and $U \subset \mathbf{R}^n$. Define $O_{i,\epsilon} = O_i/\epsilon$ for $i = 0, \ldots, M$, where O_i are from (4.8). Assuming (4.17) is false, by (4.8) and (4.15) we may define $0 < t_1 \le t_2 \le \ldots \le t_M$ by

$$t_i = \min\{t \mid \|\eta(t)\|_{V_\epsilon;O_{i,\epsilon}^{\mathbf{C}}} = r_0\}. \tag{4.19}$$

We claim that for all $i = 1, \ldots, M$,

$$\|\eta(t_i)\|_{V_\epsilon;O_{i,\epsilon}\setminus O_{i-1,\epsilon}} > 2r_0. \tag{4.20}$$

To prove, first note that by definition of t_i,

$$0 \le \left(\frac{d}{dt}\|\eta\|^2_{V_\epsilon;O_{i,\epsilon}^{\mathbf{C}}}\right)_{t=t_i} = 2\left((\eta, \frac{d\eta}{dt})_{V_\epsilon;O_{i,\epsilon}^{\mathbf{C}}}\right)_{t=t_i} = -2(\eta(t_i), W_{V_\epsilon}(\eta(t_i)))_{V_\epsilon;O_{i,\epsilon}^{\mathbf{C}}}, \tag{4.21}$$

where W_{V_ϵ}, the gradient of I^ϵ, is defined in (3.11). For $u \in E$, define $\mathcal{F}(u) \in E$ by $(\mathcal{F}(u), w)_{V_\epsilon} = \int_{\mathbf{R}^n} f(u) w \, dx$ for all $w \in E$. Then $W_{V_\epsilon}(u) = u - \mathcal{F}(u)$ for all $u \in E$. Let $z = \eta(t_i)$, and suppose $\|z\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} \leq 2r_0$. Then $\|z\|_{V_\epsilon; O^{\mathbf{C}}_{i-1,\epsilon}} \leq 3r_0$ by definition of t_i. Let $\varphi \in C^\infty(\mathbf{R}^n, \mathbf{R})$ with $\varphi \equiv 0$ on $O_{i-1,\epsilon}$, $\varphi \equiv 1$ on $O^{\mathbf{C}}_{i,\epsilon}$, and $\|\nabla \varphi\|_{L^\infty(\mathbf{R}^n)} < 2\epsilon/\rho < r_0/4C_2$. This is possible by (4.8) and (4.16). Then,

$$
\begin{aligned}
(4.22) \qquad \|\mathcal{F}(z)\|_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} &= \|\varphi \mathcal{F}(z)\|_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} \leq \|\varphi \mathcal{F}(z)\|_{V_\epsilon} = \sup_{\|w\|_{V_\epsilon} \leq 1} (\varphi \mathcal{F}(z), w)_{V_\epsilon} \leq \\
&\leq \sup_{\|w\|_{V_\epsilon} \leq 1} \left[(\mathcal{F}(z), \varphi w)_{V_\epsilon} + \frac{r_0}{4C_2} \|\mathcal{F}(z)\|_{W^{1,2}(\mathbf{R}^n)} \|w\|_{W^{1,2}(\mathbf{R}^n)} \right] \leq \\
&\text{(by Lemma 3.10)} \\
&\leq \frac{r_0}{4C_2} \|\mathcal{F}(z)\|_{W^{1,2}(\mathbf{R}^n)} + \sup_{\|w\|_{V_\epsilon} \leq 1} (\mathcal{F}(z), \varphi w)_{V_\epsilon} \leq \\
&\leq \frac{r_0}{4C_2} \|\mathcal{F}(z)\|_{V_\epsilon} + \sup_{\|w\|_{V_\epsilon} \leq 1} \int_{\mathbf{R}^n} f(z) \varphi w \leq \\
&\text{(by } V_- \geq 1) \\
&\leq r_0/4 + \sup_{\|w\|_{V_\epsilon} \leq 1} \int_{O^{\mathbf{C}}_{i-1,\epsilon}} |f(z)||w| \leq r_0/4 + r_0/4 = r_0/2.
\end{aligned}
$$

In the last line we used (4.13)-(4.14) on the first part. On the second part we used (4.16) and (4.8) to show that $O^{\mathbf{C}}_{i-1,\epsilon}$ belongs to the family of sets Θ described before (4.4). Then we invoked (4.4). (4.22) implies

$$
\begin{aligned}
(4.23) \quad (z, W_{V_\epsilon}(z))_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} &= (z, z - \mathcal{F}(z))_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} = \|z\|^2_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} - (z, \mathcal{F}(z))_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} \geq \\
&\geq \|z\|^2_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} - \|z\|_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} \|\mathcal{F}(z)\|_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}} = r_0 (r_0 - \|\mathcal{F}(z)\|_{V_\epsilon; O^{\mathbf{C}}_{i,\epsilon}}) \geq r_0 (r_0 - r_0/2) > 0.
\end{aligned}
$$

This contradicts (4.21). (4.20) is proven.

For $i = 1, \ldots, M$, define

$$
(4.24) \qquad s_i = \max\{ s < t_i \mid \|\eta(s)\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} = r_0 \}.
$$

By the definition of t_i, and (4.20), s_i is well-defined, with $t_{i-1} < s_i < t_i$. For all $i = 1, \ldots, M$ and $t \in [s_i, t_i]$,

$$
(4.25) \qquad \|\eta(t)\|_{V_\epsilon; (O \setminus O_{in})/\epsilon} \geq \|\eta(t)\|_{V_\epsilon; O_{M,\epsilon} \setminus O_{0,\epsilon}} \geq \|\eta(t)\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} \geq r_0,
$$

so $\|I'(\eta(t))\|_{V_\epsilon} > \delta_0$ by (4.6). Also, for all $i = 1, \ldots, M$, (4.20) and (4.24) give

$$
\begin{aligned}
(4.26) \qquad \|\eta(t_i) - \eta(s_i)\|_{V_\epsilon} &\geq \|\eta(t_i) - \eta(s_i)\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} \geq \\
&\geq \|\eta(t_i)\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} - \|\eta(s_i)\|_{V_\epsilon; O_{i,\epsilon} \setminus O_{i-1,\epsilon}} \geq 2r_0 - r_0 = r_0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4.27) \qquad C(b_2) &> I^\epsilon(\eta(0)) \geq I^\epsilon(\eta(t_0)) \geq I^\epsilon(\eta(t_0)) - I^\epsilon(\eta(t_M)) = \\
&= -\int_{t_0}^{t_M} \frac{d}{dt} I^\epsilon(\eta) = \int_{t_0}^{t_M} \|I^{\epsilon\prime}(\eta)\|^2_{V_\epsilon} \geq \sum_{i=1}^M \int_{s_i}^{t_i} \|I^{\epsilon\prime}(\eta)\|^2_{V_\epsilon} \geq \\
&\geq \delta_0 \sum_{i=1}^M \int_{s_i}^{t_i} \|I^{\epsilon\prime}(\eta)\|_{V_\epsilon} = \delta_0 \sum_{i=1}^M \int_{s_i}^{t_i} \|\frac{d\eta}{dt}\|_{V_\epsilon} \geq \delta_0 \sum_{i=1}^M \|\eta(t_i) - \eta(s_i)\|_{V_\epsilon} \geq r_0 \delta_0 M.
\end{aligned}
$$

This is impossible by definition of M ((4.7)). (4.17) is proven.

To complete the existence portion of the proof of Theorem 1.7, Case III, let $t_m \to \infty$ with $t_{m+1} - t_m \to 0$ as $m \to \infty$. Let $u_m = \eta(t_m)$. Then $I^{\epsilon\prime}(u_m) \to 0$ and $\|u_{m+1} - u_m\| \to 0$ as $m \to \infty$.

Define $Hull(V_\epsilon)$ to be the set of all limit points, under the uniform norm, of sequences of the form $(\tau_{x_m} V_\epsilon)$. Let ϵ be small enough so that if $\mathcal{V} \in Hull(V_\epsilon)$, $\mathcal{V}(0) \geq b_0$, and v is a nonzero critical point of $I[\mathcal{V}]$ with $\|v\|_{W^{1,2}(B_1(0))} \geq r_0$, then

$$I[\mathcal{V}](v) \geq \frac{4}{5} C(b_0). \tag{4.28}$$

It is easy to show this is possible, using Proposition 3.9.

If we apply Proposition 3.7 to any subsequence of (u_m), we obtain $k, (x^i_m)$, and (u^i_∞) as in Proposition 3.7(i)-(v). Assume that all the u^i_∞'s are "normalized" so $\|u^i_\infty\|_{W^{1,2}(B_1(0))} > r_0$ (see (4.5)) for all i. Then by (4.17), $x^i_m \in N_1(O/\epsilon)$ for all i, for large enough m, so $I^i_\infty(v_i) \geq \frac{4}{5}C(b_0)$ for all i by (4.28) and (4.0), and $\lim_{m\to\infty} I^\epsilon(u_m) \geq \frac{4}{5}kC(b_0)$. Since $I^\epsilon(u_m) < C(b_2) < \frac{3}{2}C(b_0)$ for all m (see (4.10), (4.1)), $k = 1$. That is, (u_m) does not "split into more than one piece".

Choose $(x_m) \subset N_1(O/\epsilon)$ with $\|u_m\|_{W^{1,2}(B_1(x_m))} > r_0$ for large m. Since $k = 1$ in the conclusion of Proposition 3.7, it is easy to verify that $|x_m - x_{m+1}|$ is bounded in m. Let $R > 0$ with $|x_m - x_{m+1}| < R$ for large m. Recall $\vec{v}$ from Theorem 1.7(III), and let $T = \sup\{\vec{u} \cdot x \mid x \in O/\epsilon,\ |\vec{u}| = 1,\ \vec{u} \perp \vec{v}\} < \infty$. Then there exists a sequence (δ_m) with $\delta_m \to 0$, a sequence $(t_m) \subset \mathbf{R}$, and a subsequence of (u_m) (also denoted (u_m)), with $t_m\vec{v}$ a δ_m-almost period of V and $|x_m - t_m\vec{v}| < R + T + 1$. Define $z_m = \tau_{-t_m\vec{v}} u_m$ and $V_m = \tau_{-t_m\vec{v}} V_\epsilon$. Then $V_m \to V_\epsilon$ locally uniformly and $I[V_m]'(z_m) \to 0$. Also $\|z_m\|_{B_{R+T+1}(y)} \geq r_0$. In fact, we may select a subsequence of (z_m) and a point $x \in B_{R+T+1}(0)$ with $\liminf_{m\to\infty} \|z_m\|_{W^{1,2}(B_1(x))} \geq r_0$. Applying Proposition 3.7 (and noting that $k = 1$ in Proposition 3.7(v) by the above argument), z_m converges strongly to a nonzero critical point z of I^ϵ, with $\|z\|_{W^{1,2}(B_1(x))} \geq r_0$ and $C(b_0) \leq I^\epsilon(z) < C(b_2)$. It is not apparent whether or not x must belong to O/ϵ.

Concentration of Solutions

We have proven the existence portion of Theorem 1.7, Case III, but not the positivity and exponential decay properties. In the proofs in this section, we may replace O by $O \cap \{y \mid V(y) < b_0 + \hat{\epsilon}\}$ for any $\hat{\epsilon} > 0$, and the proof works the same. Thus for any sequence (ϵ_m) with $\epsilon_m \to 0$, we can obtain a sequence (u_m) of critical points of I^{ϵ_m}, with $I^{\epsilon_m}(u_m) \to C([b_0])$, and points $x_m \in \mathbf{R}^n$ with $V(x_m) \to b_0$ and $\|u_m\|_{W^{1,2}(B_1(x_m/\epsilon_m))} \geq r_0$. By Proposition 3.9, $(\tau_{-x_m/\epsilon_m} u_m)$ converges along a subsequence in $W^{1,2}(\mathbf{R}^n)$ to a solution ω of $-\Delta\omega + V(b_0)\omega = f(\omega)$ with $I[b_0](\omega) = C[b_0]$. By arguments of [CR1], ω does not change sign. Without loss of generality, ω is positive. By elliptic theory, u_m converges pointwise to ω. A maximum principle argument shows that for large m, u_m is nonnegative. Namely, if u_m is negative somewhere for large enough m, it has a negative global minimum at $y \in \mathbf{R}^n$, with $|u_m(y)|$ small. However, the equation $-\Delta u + V(\epsilon_m x)u = f(u)$ implies that $\Delta u_m(y) > 0$ (if m is chosen to make $|u_m(y)|$ small enough). Thus $u_m \geq 0$. A slight refinement of the above argument shows that in fact $u_m > 0$.

Arguments of [GNN1-2] show that ω above has a unique local maximum, which is nondegenerate. Since $(\tau_{-x_m/\epsilon_m} u_m)$ converges to ω locally in C^2, a similar maximum principle argument, found in [FdP1], shows that u_m has a unique local maximum $\bar{x}_m$ for large m. We have shown that there exists a sequence $(x_m) \subset \mathbf{R}^n$ with $\liminf_{m\to\infty} \|u_m\|_{W^{1,2}(B_1(x_m))} > 0$. It is clear that $|x_m - \bar{x}_m|$ is bounded in m. Therefore, since $V_{\epsilon_m}(x_m) \to b_0$, $V_{\epsilon_m}(\bar{x}_m) \to b_0$ as well. The exponential decay of u_m is given by a standard maximum principle argument found in [FdP1]. The proof of Theorem 1.7 is complete. Note that is not apparent whether $\bar{x}_m$ belongs to O/ϵ_m; therefore in the statement of Theorem 1.7, it is not apparent whether z_ϵ belongs to O.

Open Questions

There are many open questions associated with (1.2) and related equations. It is unknown whether the ϵ is really necessary in (1.2) or in similar elliptic PDE containing almost periodic terms, even in special cases such as $n = 2$, or V quasiperiodic rather than merely almost periodic. It is also unclear whether the convexity condition (f_5) is necessary in equations like (1.2), even when V satisfies stronger global conditions than $(V_1) - (V_3)$.

References

[ACM] F. Alessio, P. Caldiroli, P. Montecchiari.: Genericity of the multibump dynamics for almost periodic Duffing-like systems. Preprint.

[Be] Besicovitch, A. S.: Almost Periodic Functions. Dover Publications, Berlin 1954

[Bo] Bohr, H.: Fasteperiodische Funktionen. Springer-Verlag, Berlin 1932

[C] Corduneanu, C.: Almost Periodic Functions. Chelsea Publishing Co., New York 1989

[CMN] Coti Zelati, C., Montecchiari, P., Nolasco, M.: Multibump solutions for a class of second order almost periodic Hamiltonian systems, to appear in Nonlinear Differential Equations and Applications.

[CR1] Coti Zelati, V., Rabinowitz, P.: Homoclinic Orbits for Second Order Hamiltonian Systems Possessing Superquadratic Potentials. Journal of the American Mathematical Society **4**, 693-627 (1991)

[CR2] Coti Zelati, V., Rabinowitz, P.: Homoclinic Type Solutions for a Semilinear Elliptic PDE on $\mathbf{R}^n$. Communications on Pure and Applied Mathematics **45**, 1217-1269 (1992)

[FdP1] Felmer, P., del Pino, M.: Local Mountain passes for semilinear elliptic problems in unbounded domains. Calc. Var. and PDE **4**, 121-137 (1996)

[FdP2] Felmer, P., del Pino, M.: Semi-classical states for nonlinear Schrödinger equations. J. Funct. Anal. **149** no. 1, 245-265 (1997)

[FdP3] Felmer, P., del Pino, M.: Multi-peak bound states for nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Analyse non linéaire **15** no. 2, 127-149 (1998)

[GNN1] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Comm. Math. Phys. **68**, 209-243 (1979).

[GNN2] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$, related properties via the maximum principle. Math. Anal. Appl. Part A. Adv. Math. Suppl. Studies **7A** 369-402.

[KT] Kabeya, Y.,Tanaka, K.: Uniqueness of Positive Radial Solutions of Semilinear Elliptic Equations in $\mathbf{R}^n$ and Séré's Non-degeneracy Condition. Preprint.

[Li] Li, Y.-Y.: On a singularly perturbed elliptic equation. To appear in Advances in Differential Equations.

[Lio] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. Annales de l'Institut Henri Poincaré **1**, 102-145 and 223-283, (1984)

[MNT] Montecchiari, P., Nolasco, M., Terracini, S.: Multiplicity of homoclinics for time recurrent second order systems. Calc. Var. and PDE **5**, no. 6, 523-555 (1997)

[R1] Rabinowitz, P. H.: A multibump construction in a degenerate setting. Calc. Var. and PDE **5**, 159-182 (1997)

[R2] Rabinowitz, P. H.: On a class of nonlinear Schrödinger equations. Z. angew Math. Phys. **43**, 270-291 (1992)

[R3] Rabinowitz, P. H.: Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conf. Ser. in Math., no. 65, Conf. Board Math. Sci., Washington, D.C.: 1986.

[S] Spradlin, G..: An Almost Periodic Function of Several Variables with no Local Minimum. Rendiconti dell'Universita' degli Studi di Trieste **28**, no. 1-2, 371-381 (1997).

[STT] Serra, E., Tarallo, M., Terracini, S.: On the existence of homoclinic solutions to almost periodic second order systems. Annales de l'Institut Henri Poincaré: Analyse Non Linéaire **13**, 783-812 (1996)

[WZ] Wang, X., Zeng, B.: On Concentration of Positive Bound States of Nonlinear Schrödinger Equations with Competing Potential Functions. SIAM J. Math. Anal. **28** no. 3, 633-655 (1997).

[Y] Yanagida, E.: Uniqueness of positive radial solutions of $-\Delta u + g(r)u = h(r)u^p$ in $\mathbf{R}^n$. Arch. Rational Mech. Anal. **15**, 257-274 (1991).

[Z] Zaidman, S.: Almost Periodic Functions in abstract spaces. Pitman Publishing Inc., 1985.